CHEGARALANMAGAN SOHADA 3-TARTIBLI TENGLAMA UCHUN BIR CHEGARAVIY MASALA

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Annotatsiya: *Maqolada chegaralanmagan sohada 3-tartibli tenglama uchun chegaraviy masala qaralgan bo'lib, masalaning yechimining yagonaligi integral energiya usulida, yechimning mavjudligini isbotlashda Fredgolm integral tenglamalari nazariyasidan foydalanilgan.*

Kalit so'zlar: *Grin funksiyasi, yuqori tartibli tenglama, Laplas operatori, yechimning yagonaligi, yechimning mavjudligi.*

Masalaning qo'yilishi. Ushbu

$$D=\{(x,y):0<x<\infty,0<y<\infty\}$$

sohada

$$\frac{\partial}{\partial x}\Delta U(x,y)+C(x,y)U_x(x,y)=0 \qquad (1)$$

tenglamani qaraymiz.

Bu yerda: $\Delta=\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}$ - Laplas operatori.

(1) tenglamaning $\overline{D}$ sohada uzluksiz shunday *U(x,y)* yechimini topingki, u quyidagi chegaraviy shartlarni qanoatlantirsin:

$$U(x,y)\big|_{x=0}=\varphi_1(y), \qquad 0\le y<\infty, \qquad (2)$$

$$U(x,y)\big|_{y=0}=\varphi_2(x), \qquad 0\le x<\infty, \qquad (3)$$

$$U_x(x,y)\big|_{x=0}=\varphi_3(\text{y}), \qquad 0\le y<\infty, \qquad (4)$$

$$\lim_{R\to\infty}U_x(x,y)=0, \qquad R^2=x^2+y^2, \qquad x>0,y>0. \qquad (5)$$

Bu yerda: *C(x,y)* va $\varphi_i(i=\overline{1,3})$ - berilgan funksiyalar va

$$\varphi_1(0)=\varphi_2(0) \qquad (6)$$

kelishuv sharti o'rinli.

Berilgan tenglamada

$$\frac{\partial}{\partial x}U=V \qquad (7)$$

belgilash kiritsak, u holda (1) tenglama quyidagi koʻrinishga keladi:

$$\Delta V + CV = 0 \tag{8}$$

(7) ga asosan quyidagi chegaraviy shartlarni olamiz:

$$V(x,y)\big|_{y=0} = \varphi_2'(x), \qquad V(x,y)\big|_{x=0} = \varphi_3(y), \tag{9}$$

$$\lim_{R\to\infty} V(x,y) = 0, \qquad R^2 = x^2 + y^2, \qquad x>0, y>0. \tag{10}$$

Masala yechimining yagonaligi.

Teorema. Agar $C(x,y) \geq 0, \qquad (x,y) \in D,$

shart bajarilsa, u holda (1) – (5) masalaning bittadan ortiq yechimi mavjud emas.

Isbot. Masala yechimining yagonaligini isbotlash uchun (1) - (5) masalaga mos bir jinsli, ya'ni

$$\Delta\left(\frac{\partial}{\partial x} U\right) + C\frac{\partial U}{\partial x} = 0 \tag{1}$$

$$U(x,y)\big|_{x=0} = 0, \qquad U(x,y)\big|_{y=0} = 0, \qquad U_x(x,y)\big|_{x=0} = 0. \tag{11}$$

masalani qaraymiz. U holda (8) – (10) masalaning bir jinsli masalasi quyidagicha boʻladi:

$$\Delta V + CV = 0, \tag{8}$$

$$V(x,y)\big|_{x=0} = 0, \qquad V(x,y)\big|_{y=0} = 0, \tag{12}$$

$$\lim_{R\to\infty} V(x,y) = 0, \quad R^2 = x^2 + y^2, \quad x>0, y>0. \tag{10}$$

Ushbu

$$D_R = \left\{(x,y) : x^2 + y^2 < R^2, x>0, y>0\right\},$$

$$\partial D_R = \left\{(x,y) : (x=0) \cup (y=0) \cup \sigma_R\right\},$$

$$\sigma_R = \left\{(x,y) : x^2 + y^2 = R^2, x>0, y>0\right\}$$

sohani qaraymiz.

D_R sohada ushbu integralni tahlil qilamiz:

$$\iint_{D_R} V(V_{xx} + V_{yy} + CV)\,dxdy = 0. \tag{13}$$

(13) da ba'zi shakl almashtirishlarni bajarib, (10), (12) chegaraviy shartlardan foydalansak, (13) tenglik quyidagi

$$\iint_{D_R} \left[(V_x)^2 + (V_y)^2 + CV^2\right] dxdy = 0. \tag{14}$$

koʻrinishga keladi.

1) Agar $C = 0$ boʻlsa, u holda (14) dan $V_x = V_y = 0$ tenglikni olamiz, bu

tengliklardan $V = const$ ekanligi kelib chiqadi. (1) tenglama $\overline{D}$ sohada uzluksiz boʻlgani uchun (11) ga asosan $V = 0$ ва (7) ga asosan $V = U_x = 0$ boʻladi.

Ma'lumki, bu tenglamaning umumiy yechimi

$$U=\overline{\phi}(\mathrm{y}) \qquad (15)$$

ko'rinishda bo'ladi. Bu yerda $\overline{\phi}(\mathrm{y})$ - ixtiyoriy noma'lum funksiya. (11) chegaraviy shartlarning biridan foydalansak, (15) tenglikdagi $\overline{\phi}(x)$ funksiya aynan nolga teng bo'lib $U(x,y)=0,\quad (x,y)\in\overline{D}$ da trivial yechimga ega ekanligini topamiz.

2) Agar $C\neq 0$ bo'lsa, $U_x=0$ bo'ladi va 1) holdagi kabi fikr yuritib, *U=0* ni hosil qilamiz.

Masala yechimining mavjudligi.

2-teorema. Agar

1) $\varphi_1(y),\ \varphi_2(x),\varphi_3(y)$ funksiyalar uzluksiz va

2) $\left|\varphi_1^{'}\right|\leq\dfrac{C_1}{y^2},\quad y\to\infty,\ |\varphi_2|\leq\dfrac{C_2}{x^2},\ |\varphi_3|\leq\dfrac{C_3}{y^2},\ y\to\infty$ shartlarni qanoatlantirsa,

u holda (1) - (5) masalaning yechimi mavjud.

Isbot. *V* funksiya (8), (10), (12) masalaning yechimi bo'lsin. Biror (x,y) nuqtani ε radiusli va markazi (x,y) nuqtada bo`lgan C_ε aylana bilan o'raymiz. Aylanadan tashqaridagi sohani D_R^ε deb belgilaymiz va bu soha uchun Grin formulasini [1] qo'llaymiz.

$$\iint\limits_{D_R^\varepsilon}\left[G(\Delta V-V\Delta G)\right]d\xi d\eta=\iint\limits_{\partial D_R}\left(G\frac{\partial V}{\partial n}-V\frac{\partial G}{\partial n}\right)ds \qquad (16)$$

Bunda: $G(z,\tau)=\dfrac{1}{2\pi}\ln\left|\dfrac{z^2-\tau^{-2}}{z^2-\tau^2}\right|$; bu yerda: *z=x+iy*, $\tau=\xi+i\eta$.

Ma'lumki, Grin funksiyasi

$$\Delta G=0 \qquad (17)$$

tenglamani va

$$G\big|_{\xi=0}=0\ ,\qquad G\big|_{\eta=0}=0 \qquad (18)$$

shartlarni qanoatlantiradi.

Integral chegarasini almashtirib,

$$\int\limits_{OA\cup\sigma_R\cup OB}\left(G\frac{\partial V}{\partial n}-V\frac{\partial G}{\partial n}\right)ds-\int\limits_{C\varepsilon}\left(G\frac{\partial V}{\partial n}-V\frac{\partial G}{\partial n}\right)ds=\iint\limits_{D_R^\varepsilon}GCU_x d\xi d\eta$$

ifodani hosil qilamiz. Ohirgi tenglikdan

$$I_1=\int\limits_{OA\cup\sigma_R\cup OB}\left(G\frac{\partial V}{\partial n}-V\frac{\partial G}{\partial n}\right)ds=\int\limits_{OA\cup OB}\left(G\frac{\partial V}{\partial n}-V\frac{\partial G}{\partial n}\right)ds+$$

$$+\int\limits_{\sigma_R}\left(G\frac{\partial V}{\partial n}-V\frac{\partial G}{\partial n}\right)ds=-\int\limits_{AO}V(0,\eta)G_\xi(x,y;0,\eta)d\eta+$$

$$+\int\limits_{OB}V(\xi,0)G_\eta(x,y;\xi,0)d\eta+\int\limits_{\sigma_R}\left(G\frac{\partial V}{\partial n}-V\frac{\partial G}{\partial n}\right)ds=$$

$$=\int\limits_0^R G_\xi\varphi_3(\eta)d\eta-\int\limits_0^R G_\eta\varphi_2^{'}(\xi)d\xi+\int\limits_{\sigma_R}\left(G\frac{\partial V}{\partial n}-V\frac{\partial G}{\partial n}\right)ds,$$

$$\mathrm{I}_1 = \int_0^R G_\xi \varphi_3(\eta) d\eta - \int_0^R G_\eta \varphi_2^{'}(\xi) d\xi + \int_{\sigma_R} \left(G \frac{\partial V}{\partial n} - V \frac{\partial G}{\partial n} \right) ds,$$

ifodani olamiz.

Bu ifodada $R \to \infty$ da limitga o'tib,

$$I_1 = \int_0^\infty G_\xi \varphi_3(\eta) d\eta - \int_0^\infty G_\eta \varphi_2^{'}(\xi) d\xi$$

tenglikni hosil qilamiz.

Endi C_ε bo'yicha

$$I_2 = \int_{C_\varepsilon} \left(G \frac{\partial V}{\partial n} - V \frac{\partial G}{\partial n} \right) ds$$

integralni hisoblaymiz. Buning uchun Grin funksiyasidan quyidagi ko'rinishda yozib olamiz:

$$G(z,\tau) = -\frac{1}{2\pi} \ln|z-\tau| + g(x,y;\xi,\eta) = -\frac{1}{2\pi} \ln\sqrt{(x-\xi)^2 + (y-\eta)^2} +$$
$$+ g(x,y;\xi,\eta) = -\frac{1}{2\pi} \ln r + g(x,y;\xi,\eta)$$

U holda

$$I_{21} = \int_{C_\varepsilon} G \frac{\partial V}{\partial n} ds = \int_{C_\varepsilon} \left[-\frac{1}{2\pi} \ln r + g \right] \frac{\partial V}{\partial n} ds =$$
$$= -\frac{1}{2\pi} \int_{C_\varepsilon} \left[\ln r \frac{\partial V}{\partial n} (P^*) 2\pi\varepsilon \right] ds + \int_{C_\varepsilon} g \frac{\partial V}{\partial n} ds.$$

Bu yerda $\frac{\partial V}{\partial n}(P^*) - \frac{\partial V}{\partial n}$ normal hosilaning C_ε aylana ustidagi o'rta qiymati. Bu ifodada $\varepsilon \to 0$ limitga o'tsak, $I_{21} = 0$.

Endi ushbu integralni qaraymiz:

$$I_{22} = \int_{C_\varepsilon} -V \frac{\partial G}{\partial n} ds = \int_{C_\varepsilon} V \frac{\partial G}{\partial n} ds = -\int_{C_\varepsilon} V \left(-\frac{1}{2\pi} \cdot \frac{1}{r} + \frac{\partial g}{\partial r} \right) ds =$$
$$= \frac{1}{2\pi\varepsilon} \int_{C_\varepsilon} V ds - \int_{C_\varepsilon} V \frac{\partial g}{\partial r} ds = \frac{1}{2\pi\varepsilon} V\left(P^{**}\right) 2\pi\varepsilon + \int_{C_\varepsilon} V \frac{\partial g}{\partial r} ds =$$
$$= V\left(P^{**}\right) + \int_{C_\varepsilon} V \frac{\partial g}{\partial r} ds.$$

Bu yerda P^{**} -chiziq ustidagi ixtiyoriy nuqta. $\varepsilon \to 0$ da limitga o'tib, $I_{22} = V(x,y)$ tenglikni hosil qilamiz. Natijada: $I_2 = V(x,y)$.

U holda (8), (10), (12) masalaning yechimini

$$V(x,y)=\int_0^\infty G_\xi\left(x,y;0,\eta\right)\varphi_3(\eta)d\eta-\int_0^\infty G_\eta\left(x,y;\xi,0\right)\varphi_2^{'}(\xi)d\xi-$$

$$-\iint_D G(x,y;\xi,\eta)C(\xi,\eta)U_x(\xi,\eta)d\xi d\eta$$

ko‘rinishda ifodalanadi.

Grin funksiyasini ko‘rinishidan foydalansak:

$$V(x,y)=\frac{1}{\pi}\int_0^\infty\left(\frac{x}{x^2+(y-\eta)^2}-\frac{x}{x^2+(y+\eta)^2}\right)\varphi_3(\eta)d\eta-$$

$$-\frac{1}{\pi}\int_0^\infty\left(\frac{y}{\left(x-\xi\right)^2+y^2}-\frac{y}{\left(x+\xi\right)^2+y^2}\right)\varphi_2^{'}(\xi)d\xi- \quad (19)$$

$$-\iint_D C(\xi,\eta)\mathrm{G}(x,y;\xi,\eta)U_x(\xi,\eta)d\xi d\eta$$

ifodaga ega bo‘lamiz. (7) belgilashga asosan

$$U_x(x,y)+\iint_D K(x,y;\xi,\eta)U_x(\xi,\eta)d\xi d\eta=F(x,y) \quad (20)$$

ko‘rinishdagi tenglamaga kelamiz. Bu yerda

$$K(x,y;\xi,\eta)=C(\xi,\eta)G(x,t;\xi,\eta)$$

$$F(x,y)=-\int_0^\infty\left(\frac{x}{x^2+(y-\eta)^2}-\frac{x}{x^2+(y+\eta)^2}\right)\varphi_3(\eta)d\eta+$$

$$+\int_0^\infty\left(\frac{y}{\left(x-\xi\right)^2+y^2}-\frac{y}{\left(x+\xi\right)^2+y^2}\right)\varphi_2^{'}(\xi)d\xi,$$

(20) tenglama Fredgolmning ikkinchi tur integral tenglamasi bo‘lib, yechimning yagonaligi mavjudlik teoremasidan kelib chiqadi.

Uni yechimi rezolventa orqali

$$U_x=\iint_D\Gamma(x,\mathrm{y};\xi,\eta)F(\xi,\eta)\,\mathrm{d}\xi\,\mathrm{d}\eta \quad (21)$$

ko‘rinishda ifodalanadi.

(21) tenglamani (2) yoki (3) shartlardan biri bilan yechib, (1) – (5) masalani yechimini topamiz.

Teorema isbotlandi.

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