MURAKKAB ARALASH TURDAGI UCHINCHI TARTIBLI BIR TENGLAMA UCHUN CHEGARAVIY MASALANING YECHIMNI MAVJUDLIGI TEOREMASI

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Annotatsiya:*Mazkur maqolada chegaralanmagan sohada uchinchi tartibli tenglama uchun chegaraviy masala qaralgan bo'lib yechimining mavjudligi Fredgolmning ikkinchi tur integral tenglamalari nazariyasidan foydalangan holda isbotlangan.*

Kalit so`zlar:*Yuqori tartibli tenglama, Fredgolm integral tenglamasi,,yechimning mavjudligi,Grin funksiyasi.*

Murakkab va murakkab-aralash tipdagi tenglamalar uchun respublikamizda ustozlarimiz T.D.Jo'rayev $[1]$, M.S.Salohiddinov $[2]$ va ularning shogirdlari tomonidan chegaraviy masalalar qo'yilib ularni o'rganish nazariyalari yaratilgan.Bu maqolada uchinchi tartibli tenglama uchun chegaralanmagan sohada chegaraviy masala o'rganilgan. Bu masala yechimining yagonaligi $[4]$da ibotlangan.

Masalaning qo'yilishi:

Ushbu

$$\begin{cases} \frac{\partial}{\partial x}(U_{xx}+U_{yy})+C(x,y)U(x,y)=0, y>0 \\ \frac{\partial}{\partial x}(U_{xx}-U_{yy})=0, y<0 \end{cases} \qquad (1)$$

tenglamani $D=\{D_1 \cup D_2 \cup y=0\}$ sohada qaraymiz.

Bunda $D_1=\{(x,y): x>0, y>0\}$ $D_2=\{(x,y): x>0, y>-x\}$, $C(x,y)$ berilgan funksiya.

Masala. D sohada (1) tenglamaning quyidagi shartlarni qanoatlantiruvchi $U(x,y)$ yechimi topilsin:

1) $U(x,y)$ funksiya $\bar{D}$ da uzluksiz;

2) $U(x,y)$ D sohada (1) tenglamaning regulyar yechimi, $y \neq 0$;

3) $U(x,y)$ quyidagi chegaraviy shartlarni qanoatlantirsin:

$$U(0,y)=\varphi_1(y), \qquad 0 \le y < \infty \qquad (2)$$

$$U_x(0,y)=\varphi_2(y), \qquad 0 \le y < \infty \qquad (3)$$

$$U(x,-x)=\psi_1(x), \quad 0\le x<\infty \qquad (4)$$

$$\frac{\partial U(x,-x)}{\partial n}=\psi_2(x), \quad 0<x<\infty \qquad (5)$$

$$\lim_{R\to\infty} U_x=0, \quad R^2=x^2+y^2, \quad x>0, \;\; y>0 \qquad (6)$$

bu yerda $\varphi_i(y),\psi_i(x),\ (i=1,2)$ berilgan funksiyalar bo'lib, $\varphi_1(0)=\psi_1(0)$ kelishuv shartini qanoatlantiradi.

$$\frac{\partial}{\partial x}U=V \qquad (7)$$

deb belgilash kiritsa, u holda (1) tenglama

$$\begin{cases}\Delta V+CU=0\\ \Box V=0\end{cases} \qquad (8)$$

ko'rinishni oladi, bunda

$$\Delta=\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}-elliptik\ \ operator,$$

$$\Box=\frac{\partial^2}{\partial x^2}-\frac{\partial^2}{\partial y^2}-giperbolik\ \ operator.$$

(2)-(6) chegaraviy shartlardan foydalanib noma'lum $V(x,y)$ funksiya uchun quyidagi chegaraviy shartlarni olamiz;

$$V(x,y)\,|_{y=-x}=\frac{1}{2}\left(\psi_1'(x)+\sqrt{2}\psi_2(x)\right)=\psi(x)\,, \qquad (9)$$

$$V(x,y)\,|_{x=0}=\varphi_2(y)\,, \qquad (10)$$

$$\lim_{R\to\infty}V(x,y)=0\,. \qquad (11)$$

MASALA YECHIMINING MAVJUDLIGI

Teorema. Agar berilgan funksiyalar

a) $|\,C(x,y)\,|\le\dfrac{N_1}{R}\,, R\to\infty\,, N_1=const\,, \quad C_x(x,y)\ge 0$

b) $\varphi_i\left(y\right),\varphi:\left(x\right)\ \left(i=1,2\right)$ funksiyalar uzluksiz;

$$\left|\varphi_2\left(y\right)\right|\le\frac{C_1}{R^2}\,,\ \ \left|\varphi_1\left(y\right)\right|\le\frac{C_2}{R^2}\,, \qquad C\left(x,y\right)\le\frac{C_3}{R^{2+\varepsilon}}\,,$$

$$\left|\psi_1\left(x\right)\right|\le\frac{C_4}{R^{\varepsilon}}\,,\ \ \left|\psi_2\left(x\right)\right|\le\frac{C_5}{R^{\varepsilon}}\,, R\to\infty$$

$\left(c_j=conts,\ \ j=\overline{1,5}\right)$ shartlarni qanoatlantirsa, u holda $\left(1\right)-\left(6\right)$ masalani yechimi mavjud.

ISBOT: $\left(7\right)$ belgilashga ko'ra $\left(1\right)$ tenglamadan D_1 sohada

$$V_{xx}+V_{yy}=-CU \qquad (12)$$

tenglamani, (3), (6), shartlardan esa(10), (11) shartlarni olamiz. D_2 sohada (1) tenglamani

$$U_{xx} - U_{yy} = \omega(y) \tag{13}$$

ko‘rinishida yozib olamiz, bunda $\omega(y)$ noma’lum funksiya. (13) ning umumiy yechimi [3]

$$U(x,y) = F(x+y) + \Phi(x-y) + \frac{1}{4}\int_0^y dy \int_{x-y+\eta}^{x+y-\eta} \omega(\eta)\, d\eta \tag{14}$$

ko‘rinishida bo’ladi. (5) shartdan foydalanib (14) dan

$$\omega(y) = \sqrt{2}\psi_2(-y) \tag{15}$$

ekanligini topamiz. (4) shartdan foydalanib va (15) ni nazarda tutsak quyidagiga ega bo‘lamiz:

$$U(x,y) = F(x+y) + \psi_1\left(\frac{x-y}{2}\right) - F(0) + P(x,y) \tag{16}$$

bu yerda

$$P(x,y) = \frac{\sqrt{2}}{4}\int_0^y dy \int_{x-y+\eta}^{x+y-\eta} \psi_2(-\eta)\, d\eta .$$

(16) dan va $U(x,0) = \tau(x)$, $U_y(x,0) = \nu(x)$ belgilashlarga ko‘ra

$$\tau'(x) - \nu(x) = \psi_0(x) \tag{17}$$

ko‘rinishidagi noma’lum funksiyalar uchun birinchi munosabatni olamiz.

Bu yerda

$$\psi_0(x) = -\psi_1'\left(\frac{x}{2}\right) + P_x(x,0) - P_y(x,0).$$

Bundan x bo`yicha xosila olib

$$\tau''(x) - \nu'(x) = \psi_0'(x) \tag{18}$$

ko‘rinishidagi munosabatga ega bo‘lamiz.

Endi D_1 sohada (12) tenglamaning (10) va $V_y(x,0) = \nu'(x)$ shartlarni qanoatlantiruvchi yechimini Grin funksiyasi orqali yozamiz [3] :

$$V(x,y) = \int_0^\infty G(x,y,\xi,0)\nu'(\xi)d\xi - \int_0^\infty G_\xi(x,y,0,\eta)\varphi_2(\eta)d\eta + \iint_{D_1} G(x,y,\xi,\eta)C(\xi,\eta)U(\xi,\eta)d\xi d\eta$$

yoki

$$G(x,y,\xi,\eta) = \frac{1}{2\pi}\ln\left|\frac{z^2 - \bar{\zeta}^2}{z^2 - \zeta^2}\right|, \quad z = x + iy, \ \zeta = \xi + i\eta$$

ko‘rinishidagi Grin funksiyasini qo‘llasak

$$V(x,y)=\frac{1}{\pi}\int_0^{\infty}\left[\frac{x}{x^2+(y-\eta)^2}+\frac{x}{x^2+(y+\eta)^2}\right]\varphi_2(\eta)d\eta-$$
$$-\frac{1}{2\pi}\int_0^{\infty}\ln\left|\frac{(x+\xi)^2+y^2}{(x-\xi)^2+y^2}\right|\nu'(\xi)d\xi+\iint_{D_1}G(x,y,\xi,\eta)C(\xi,\eta)U(\xi,\eta)d\xi d\eta \qquad (19)$$

formulaga ega bo‘lamiz. (19) da $y=0$ deb x bo`yicha xosila olamiz

$$V_x(x,0)=-\frac{1}{\pi}\int_0^{\infty}\left[\frac{1}{\xi-x}+\frac{1}{\xi+x}\right]\nu'(\xi)d\xi+$$
$$+\frac{2}{\pi}\int_0^{\infty}\frac{\eta^2-x^2}{(x^2+\eta^2)^2}\varphi_2(\eta)d\eta+\iint_{D_1}G_x(x,0,\xi,\eta)C(\xi,\eta)U(\xi,\eta)d\xi d\eta.$$

$V_x(x,0)=\tau''(x)$ ekanligini hisobga olsak va (18) ga ko‘ra

$$\nu'(x)+\frac{1}{\pi}\int_0^{\infty}\left[\frac{1}{\xi-x}+\frac{1}{x+\xi}\right]\nu'(\xi)d\xi-\iint_{D_1}G_x(x,0,\xi,\eta)C(\xi,\eta)U(\xi,\eta)d\xi d\eta=F_1(x)$$

(20)

bu yerda

$$F_1(x)=\frac{2}{\pi}\int_0^{\infty}\frac{\eta^2-x^2}{\left(x^2+\eta^2\right)^2}\varphi_2(\eta)d\eta-\psi_0'(x). \qquad (21)$$

(20) tenglamaga ushbu

$$Kf=f(x)+\frac{1}{\pi}\int_0^{\infty}\left[\frac{1}{\xi-x}-\frac{1}{\xi+x}\right]f(\xi)d\xi$$

operatorni ta’sir ettirsak, unga ekvivalent bo‘lgan [5]

$$\frac{1}{\pi}\int_0^{\infty}\frac{\nu'(\xi)}{\xi-x}=F_2(x) \qquad (22)$$

tenglamani olamiz. Bunda

$$F_2(x)=\frac{1}{2}K\overline{F}_1(x)\ ,$$
$$\overline{F}_1(x)=F_1(x)-\iint_{D_1}G_x(x,0,\xi,\eta)C(\xi,\eta)U(\xi,\eta)d\xi d\eta.$$

(22) tenglamani $x=0$ da birdan kichik maxsuslikka ega va $x\to\infty$ da chegaralangan yechimi [6]

$$\nu'(x)=-\frac{1}{\pi\sqrt{x}}\int_0^{\infty}\sqrt{\xi}\,\frac{F_2(\xi)}{\xi-x}d\xi \qquad (23)$$

ko‘rinishda ifodalanadi. $F_2(\xi)$ ning ifodasini qo‘ysak

$$\nu'(x) = -\frac{1}{2\pi\sqrt{x}}\int_0^\infty \sqrt{\xi_1}\,\frac{1}{\xi_1 - x}\left[F_1(\xi_1) - \iint_{D_1} G(\xi_1,0,\xi,\eta)C(\xi,\eta)U(\xi,\eta)d\xi d\eta\right]d\xi -$$

$$-\frac{1}{2\pi\sqrt{x}}\int_0^\infty \frac{\sqrt{\xi_1}}{\xi_1 - x}\left[\int_0^\infty\left(\frac{1}{\bar{\xi}-\xi_1}+\frac{1}{\bar{\xi}+\xi_1}\right)\left[F_1(\bar{\xi}) - \iint_{D_1} G_x(\bar{\xi},0,\xi,\eta)C(\xi,\eta)U(\xi,\eta)d\xi d\eta\right]d\bar{\xi}\right]d\xi_1.$$

Endi (19) formuladan foydalanib (7) tenglamani (2) shart ostida yechamiz.

$$U = \int_0^x\left[\int_0^\infty\left[\frac{t}{t^2+(y-\eta)^2}+\frac{t}{t^2+(y+\eta)^2}\right]\varphi_2(\eta)d\eta\right] -$$

$$-\frac{1}{2\pi}\int_0^\infty \ln\left|\frac{(t+\xi)^2+y^2}{(t-\xi)^2+y^2}\right|\nu'(\xi)d\xi +$$

$$+\iint_{D_1}\left[\quad G(t,y,\xi,\eta)CUd\xi d\eta\quad\right]dt + \varphi_1(y).$$

Bu yerda matematik analizning xosmas integrallar to'g'risidagi teoremasiga asoslanib integrallash tartibini o'zgartiramiz va t bo'yicha integrallaymiz.

$$I_1 = \int_0^x\left[\frac{t}{t^2+(y-\eta)^2}+\frac{t}{t^2+(y+\eta)^2}\right]dt = \frac{1}{2}\ln\left|\frac{x^2+(y-\eta)}{(y-\eta)^2}\right| + \frac{1}{2}\ln\left|\frac{x^2+(y+\eta)^2}{(y+\eta)^2}\right|$$

,

$$I_2 = \int_0^x \ln\left|(t+\xi)^2+y^2\right|dt - \int_0^x \ln\left|(t-\xi)^2+y^2\right|dt = (x+\xi)\ln\left|(x+\xi)^2+y^2\right| - (x-\xi)\ln\left|(x-\xi)^2+y^2\right| +$$

$$+2y\,arctg\frac{x+\xi}{y} - 2y\,arctg\frac{x-\xi}{y} - 2\xi\ln\left(\xi^2+y^2\right) - 4x + 4\xi.$$

U holda quyidagi tenglamaga kelamiz.

$$U(x,y) - \iint_{D_1} \underline{K}U(\xi,\eta)d\xi d\eta + \int_0^\infty K_2\nu'(\xi)d\xi = F_2(x,y). \qquad (24)$$

Bunda

$$\underline{K}(x,y,\xi,\eta) = C(\xi,\eta)\int_0^x G(t,y,\xi,\eta)dt \ ,$$

$$K_2(x,y,\xi) = (x+\xi)\ln\left|(x+\xi)^2+y^2\right| - (x-\xi)\ln\left|(x-\xi)^2+y^2\right| - 4x + 2y\,arctg\frac{x+\xi}{2} -$$

$$-2y\,arctg\frac{x-\xi}{2} - 2\xi\ln\left|\xi^2+y^2\right| + 4\xi \ ,$$

$$F_2(x,y) = \int_0^\infty K_1(x,y,\eta)\varphi_2(\eta)d\eta + \varphi_1(y) \ ,$$

$$K_1(x,y,\eta)=\frac{1}{2}\ln\left|\frac{x^2+(y+\eta)^2}{(y+\eta)^2}\right|+\frac{1}{2}\ln\left|\frac{x^2+(y-\eta)^2}{(y-\eta)^2}\right|$$

oxirgi tenglamaga $\nu'(x)$ ning (23) dagi ifodasini qo'yamiz.

$$U(x,y)-\iint_{D_1}\underline{K}(x,y,\xi,\eta)U(\xi,\eta)d\xi d\eta+\iint_{D_1}\left[\int_0^\infty K_2(x,y,\xi_1)K_3(\xi_1,\xi,\eta)d\xi_1\right]U(\xi,\eta)d\xi d\eta=$$

$$=F_2(x,y)-\int_0^\infty K_2(x,y,\xi)F_0(\xi)d\xi$$

yoki

$$U(x,y)-\iint_{D_1}K_4(x,y,\xi,\eta)U(\xi,\eta)d\xi d\eta=F_{20}(x,y) \qquad (25)$$

bunda

$$K_4(x,y,\xi,\eta)=K(x,y,\xi,\eta)+\int_0^\infty K_2(x,y,\xi_1)K_3(\xi_1,\xi,\eta)d\xi_1\ ,$$

$$F_{20}(x,y)=F_2(x,y)-\int_0^\infty K_2(x,y,\xi)F_0(\xi)d\xi,$$

$$K_3(x,\xi,\eta)=\frac{1}{\sqrt{x}}\int_0^\infty G_x(\xi_1,0,\xi,\eta)d\xi_1-\int_0^\infty\frac{\sqrt{\xi_1}}{\xi_1-x}d\xi_1\int_0^\infty\left[\frac{1}{\bar{\xi}-\xi_1}+\frac{1}{\bar{\xi}+\xi_1}\right]G_x(\bar{\xi},0,\xi,\eta)d\bar{\xi}\ ,$$

$$F_0(x)=-\frac{1}{\pi\sqrt{x}}\int_0^\infty\frac{\sqrt{\xi_1}\,F_1(\xi_1)}{\xi_1-x}d\xi_1+\frac{1}{\pi\sqrt{x}}\int_0^\infty\frac{\sqrt{\xi_1}}{\xi_1-x}d\xi_1\int_0^\infty\frac{1}{\bar{\xi}-\xi_1}+\frac{1}{\bar{\xi}+\xi_1}\Bigg]F_1(\bar{\xi})d\bar{\xi}$$

(25) tenglama Fredgolm II tur tenglamasi bo'lib, yechimning mavjudligi yagonalik teoremasidan kelib chiqadi. Mavjudlik teoremasi shartlari bajarilganda $U(x,y)$ funksiya masalaning barcha shartlarini qanoatlantiradi.

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