

SELECTED METHODS FOR SOLVING LOGARITHMIC INEQUALITIES

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Annotation: *This article discusses methods for solving logarithmic inequalities by reducing them to simple algebraic equations and inequalities. Relevant examples are provided.*

Keywords: *logarithmic equation, inequality, variable substitution, solving systems, first factor, Vieta's theorem.*

Annotatsiya: *Ushbu maqolada logarifmik tengsizliklarni yechishda sodda algebraik tenglama va tengsizliklarga keltirib yechiladigan usullariga doir misollar keltirib o'tilgan.*

Kalit so'zlar: *Logarifmik tenglama, tengsizlik, belgilash kirish, sistemalarni yechish, birinchi ko'paytuvchi, Viyet teoremasi.*

Аннотация: *В данной статье рассмотрены методы решения логарифмических неравенств, сводящихся к простым алгебраическим уравнениям и неравенствам. Приведены соответствующие примеры.*

Ключевые слова: *логарифмическое уравнение, неравенство, введение обозначений, решение систем, первый множитель, теорема Виета.*

In the process of studying mathematics in greater depth and solving Olympiad-level problems, school students frequently encounter logarithmic equations and inequalities. In addition, solving various applied problems often leads to dealing with such types of equations.

Generally, there exist established methods and algorithms for solving logarithmic equations and inequalities. However, certain types can be transformed into simpler algebraic equations and inequalities through elementary substitutions.

Furthermore, in some logarithmic inequalities, the variable may appear in the base of the logarithm, and the expression may also involve trigonometric or even irrational components. In such cases, solving the inequality using general methods often requires additional conditions and justifications, which can make the solution process significantly longer.

In this paper, we present several examples that demonstrate alternative approaches to solving such problems with fewer constraints [1–4].

Example 1. Solve the inequality $\log_{x+\frac{1}{x}}\left(x^2 + \frac{1}{x^2} - 4\right) \geq 1$.

Solution: It is known that for any $x > 0$, the inequality $x + \frac{1}{x} \geq 2$ holds. Therefore,

there is no need to require that the base of the given inequality be greater than one. In turn, we can rewrite the given inequality as:

$$x^2 + \frac{1}{x^2} - 4 \geq x + \frac{1}{x} \text{ or } \left(x + \frac{1}{x}\right)^2 - \left(x + \frac{1}{x}\right) - 6 \geq 0.$$

By applying Vieta's Theorem to the latter inequality, we transform it into the following form and solve it:

$$\left(x + \frac{1}{x} - 3\right)\left(x + \frac{1}{x} + 2\right) \geq 0.$$

The value of the second factor in the resulting product is positive, since $x + \frac{1}{x} \geq 2$,

meaning that the positivity of the product depends solely on the first factor:

$x + \frac{1}{x} - 3 \geq 0$, where $x > 0$. We multiply both sides of this inequality by xxx and solve it:

$$x^2 - 3x + 1 \geq 0, \quad x > 0 \text{ or } \left(x - \frac{3 + \sqrt{5}}{2}\right)\left(x - \frac{3 - \sqrt{5}}{2}\right) \geq 0, \quad x > 0.$$

Thus, we obtain the solution $0 < x \leq \frac{3 - \sqrt{5}}{2}, x \geq \frac{3 + \sqrt{5}}{2}$.

Example 2. Solve the inequality

$$\log_{\frac{1}{2}}(\sin 4x + 2) + \log_2 5 \geq 2 - \log_{\frac{1}{2}} \sin 4x.$$

Solution: For this inequality, it is sufficient that $\sin 4x > 0$. We can rewrite the inequality in the following form:

$$\log_{\frac{1}{2}}(\sin 4x + 2) + \log_{\frac{1}{2}} \sin 4x \geq \log_{\frac{1}{2}} \frac{1}{4} + \log_{\frac{1}{2}} 5$$

or

$$\log_{\frac{1}{2}}((\sin 4x + 2)\sin 4x) \geq \log_{\frac{1}{2}} \frac{5}{4}.$$

This leads to the inequality:

$$\sin^2 4x + 2\sin 4x - \frac{5}{4} \leq 0.$$

In the resulting inequality, we introduce the substitution $\sin 4x = t$, which transforms it into a simple quadratic inequality that we can solve:

$$4t^2 + 8t - 5 \leq 0.$$

Thus, we obtain:

$$-\frac{5}{2} \leq t \leq \frac{1}{2} \quad \text{or} \quad -\frac{5}{2} \leq \sin 4x \leq \frac{1}{2},$$

meaning that

$$0 < \sin 4x \leq \frac{1}{2}.$$

The solution is:

$$\frac{k\pi}{2} < x \leq \frac{\pi}{24} + \frac{k\pi}{2}, \quad \frac{5\pi}{24} + \frac{k\pi}{2} < x \leq \frac{\pi}{4} + \frac{k\pi}{2}.$$

Example 3. Solve the inequality $\log_{2\sin x} \sqrt{1 + 2\cos 2x} < \frac{1}{2}$.

Solution: The problem of solving the given inequality reduces to solving the following systems:

$$A) \left\{ \begin{array}{l} \sin x > \frac{1}{2} \\ 0 < \sqrt{1 + 2\cos 2x} < \sqrt{2\sin x} \end{array} \right\}$$

$$B) \left\{ \begin{array}{l} 0 < \sin x < \frac{1}{2} \\ \sqrt{1 + 2\cos 2x} > \sqrt{2\sin x} \end{array} \right\}$$

For case A:

We obtain the inequalities $\sin x > \frac{1}{2}$, $\sin^2 x < \frac{3}{4}$ and $4\sin^2 x + 2\sin x - 3 > 0$

, and we find their solutions:

$$\sin x > \frac{1}{2}, \quad \left(\sin x + \frac{1 + \sqrt{13}}{4}\right)\left(\sin x - \frac{\sqrt{13} - 1}{4}\right) > 0.$$

From this, we have:

$$\frac{\sqrt{13} - 1}{4} < \sin x < \frac{\sqrt{3}}{2} \quad \text{or} \quad 2k\pi + \arcsin \frac{\sqrt{13} - 1}{4} < x < 2k\pi + \frac{\pi}{3}, \quad \text{and the}$$

$$\text{solutions are } 2k\pi + \frac{2\pi}{3} < x < 2k\pi + \pi - \arcsin \frac{\sqrt{13} - 1}{4}.$$

For case B:

We obtain the inequalities $0 < \sin x < \frac{1}{2}$, $4\sin^2 x + 2\sin x - 3 < 0$. From this,

we have:

$$0 < \sin x < \frac{1}{2}, \quad \sin x < \frac{\sqrt{13}-1}{4},$$

meaning that:

$$0 < \sin x < \frac{1}{2}, \quad 2k\pi < x < 2k\pi + \frac{\pi}{6}, \text{ and } 2k\pi + \frac{5\pi}{6} < x < (2k+1)\pi.$$

Example 4. Solve the inequality $(\log_{\sin x} 2)^2 < \log_{\sin x} (4\sin^2 x) + 1$.

Solution: It is known that for the sine function, $|\sin x| \leq 1$ is valid. Therefore, it is sufficient to solve the inequality in the form of $0 < \sin x < 1$. Now, we can rewrite the inequality as:

$$(\log_{\sin x} 2)^2 - 2\log_{\sin x} 2 - 3 < 0 \text{ or } (\log_{\sin x} 2 - 3)(\log_{\sin x} 2 + 1) < 0.$$

In the resulting inequality, considering that $0 < \sin x < 1$, $\log_{\sin x} 2 < 0$, we conclude that $\log_{\sin x} 2 - 3 < 0$.

Since the first factor in the product is negative and the second factor is positive, we have: $\log_{\sin x} 2 > -1$. We solve the resulting inequality:

$$\log_{\sin x} 2 > \log_{\sin x} (\sin x)^{-1} \text{ or } (\sin x)^{-1} > 2, \quad 0 < \sin x < \frac{1}{2}.$$

Thus, the solutions are:

$$2k\pi < x < 2k\pi + \frac{\pi}{6} \text{ and } 2k\pi + \frac{5\pi}{6} < x < (2k+1)\pi.$$

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