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STRANGE INEQUALITIES

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Abstract: *Inequalities and their calculations are a difficult task even for the brightest minds in our society. We struggle to find solutions to them. However, as mathematicians, we constantly strive to solve these difficult problems and create simple solutions. The inequalities encountered in the International Mathematical Olympiads are examples and problems that encourage not only students but also teachers to seek knowledge and work tirelessly on themselves.*

Key words: *Inequality, identity, equality, vector, scalar product, proof.*

INTRODUCTION

Mathematics deals with the spatial forms of objects in the material world and the quantitative relationships between them. We constantly strive to understand these relationships and rely on the scientific method as the main tool for research. These research methods are also fundamental in teaching mathematics. In this article, we will try to simplify complex inequalities by analyzing them as simply as possible. Since many students have difficulty analyzing and solving examples of inequalities in the International Mathematical Olympiads, we will provide them with several examples.

DISCUSSION AND RESULTS

When analyzing mathematical inequalities, every student thinks that the same method or only the same method is always used, unfortunately, this is a mistake. We have tried to analyze and solve some of the inequalities that are used in International Mathematical Olympiads.

Example 1.

For positive numbers a+b+c+d=1, prove that the inequality $\frac{a^2}{a+b}+\frac{b^2}{b+c}+\frac{c^2}{c+d}+\frac{d^2}{a+d}\geq\frac{1}{2}$ holds.

Solution: $\frac{a^2}{a+b}+\frac{a+b}{4}+\frac{a}{2}\geq 3\sqrt[3]{\frac{a^2}{a+b}*\frac{a+b}{4}*\frac{a}{2}}=\frac{3a}{2}$

$$\frac{b^2}{b+c}+\frac{b+c}{4}+\frac{b}{2}\geq 3\sqrt[3]{\frac{b^2}{b+c}*\frac{b+c}{4}*\frac{b}{2}}=\frac{3b}{2}$$

$$\frac{c^2}{c+d}+\frac{c+d}{4}+\frac{c}{2}\geq 3\sqrt[3]{\frac{c^2}{c+d}*\frac{c+d}{4}*\frac{c}{2}}=\frac{3c}{2}$$

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$$\frac{d^2}{a+d}+\frac{a+d}{4}+\frac{d}{2}\geq 3\sqrt[3]{\frac{d^2}{a+d}*\frac{a+d}{4}*\frac{d}{2}}=\frac{3d}{2}$$

Now add the right side to right and the left side to the left of the inequalities:

$$\frac{a^2}{a+b}+\frac{a+b}{4}+\frac{a}{2}+\frac{b^2}{b+c}+\frac{b+c}{4}+\frac{b}{2}+\frac{c^2}{c+d}+\frac{c+d}{4}+\frac{c}{2}+\frac{d^2}{a+d}+\frac{a+d}{4}+\frac{d}{2}\geq\frac{3}{2}(a+b+c+d)$$

It comes out:

$$\frac{a^2}{a+b}+\frac{b^2}{b+c}+\frac{c^2}{c+d}+\frac{d^2}{a+d}+a+b+c+d\geq\frac{3}{2}(a+b+c+d)$$

It’s proved that:

$$\frac{a^2}{a+b}+\frac{b^2}{b+c}+\frac{c^2}{c+d}+\frac{d^2}{a+d}\geq\frac{1}{2}$$

Example 2:

Prove the inequality:

$$\frac{1}{a^3+b^3+abc}+\frac{1}{b^3+c^3+abc}+\frac{1}{a^3+c^3+abc}\leq\frac{1}{abc}$$ (a,b,c- positive numbers)

Solution:

$$\frac{abc}{a^3+b^3+abc}+\frac{abc}{b^3+c^3+abc}+\frac{abc}{a^3+c^3+abc}\leq 1$$

To solve this inequality, we use an additional inequality

$(a-b)^2\geq 0$

$a^2-2ab+b^2\geq 0$

$a^2-ab+b^2\geq ab$

$a^2-ab+b^2\geq ab \quad /*(a+b)$

$(a+b)(a^2-ab+b^2)\geq ab(a+b)$

$a^3+b^3\geq ab(a+b)$

From this $\frac{abc}{a^3+b^3+abc}\leq\frac{abc}{ab(a+b)+abc}=\frac{c}{a+b+c}$

$\frac{abc}{b^3+c^3+abc}\leq\frac{abc}{bc(b+c)+abc}=\frac{a}{a+b+c}$

$\frac{abc}{a^3+c^3+abc}\leq\frac{abc}{ac(a+c)+abc}=\frac{b}{a+b+c}$

Now we add the right side of the inequalities to the right and the left side to the left:

$$\frac{abc}{a^3+b^3+abc}+\frac{abc}{b^3+c^3+abc}+\frac{abc}{a^3+c^3+abc}\leq\frac{c}{a+b+c}+\frac{a}{a+b+c}+\frac{b}{a+b+c}$$

$$\frac{abc}{a^3+b^3+abc}+\frac{abc}{b^3+c^3+abc}+\frac{abc}{a^3+c^3+abc}\leq 1 \quad /:abc$$

It is proved that:

$$\frac{1}{a^3+b^3+abc}+\frac{1}{b^3+c^3+abc}+\frac{1}{a^3+c^3+abc}\leq\frac{1}{abc}$$

Example 3:

Find the smallest value of the expression.

$x+y+\frac{2}{x+y}+\frac{1}{2xy}\ (x>0\,,y>0)$

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Solution: We can write the expression as follows:

$$\frac{x+y}{2}+\frac{2}{x+y}+\frac{1}{2xy}+\frac{x+y}{2}$$

From this: We apply Cauchy's inequality:

$$\frac{x+y}{2}+\frac{2}{x+y}\geq 2\sqrt{\frac{x+y}{2}*\frac{2}{x+y}}=2$$

$$\frac{1}{2xy}+\frac{x}{2}+\frac{y}{2}\geq 3\sqrt[3]{\frac{1}{2xy}*\frac{x}{2}*\frac{y}{2}}=\frac{3}{2}$$

Now we add the right side of the inequalities to the right and the left side to the left:

$$\frac{x+y}{2}+\frac{2}{x+y}+\frac{1}{2xy}+\frac{x}{2}+\frac{y}{2}\geq 2+1.5$$

$$x+y+\frac{2}{x+y}+\frac{1}{2xy}\geq 3.5$$

The answer is: 3.5

Example 4:

Prove the inequality: $\frac{a}{b+c}+\frac{b}{a+c}+\frac{c}{a+b}\geq\frac{3}{2}$

Solution: Method 1: introduce additional equalities X=$\frac{a}{b+c}+\frac{b}{a+c}+\frac{c}{a+b}$, Y=$\frac{b}{b+c}+\frac{a}{a+c}+\frac{b}{a+b}$,

Z=$\frac{c}{b+c}+\frac{c}{a+c}+\frac{a}{a+b}$

Here Y+Z=3 and Y+X≥3 , Z+X≥3. Now we add the right side of the inequalities to the left and the left side to the right: Y+Z+2X≥6 hence 2X≥3 and X≥3/2

It is proved that: $\frac{a}{b+c}+\frac{b}{a+c}+\frac{c}{a+b}\geq\frac{3}{2}$

Method 2: Let b+c=x,a+c=y,a+b=z

Then $a=\frac{y+z-x}{2}$, $b=\frac{x+z-y}{2}$, $c=\frac{x+y-z}{2}$ now we change the form of the above inequality in terms of data

$\frac{y+z-x}{2x}+\frac{x+z-y}{2y}+\frac{x+y-z}{2z}\geq\frac{3}{2}$

$$\frac{y}{x}+\frac{z}{x}-1+\frac{x}{y}+\frac{z}{y}-1+\frac{x}{z}+\frac{y}{z}-1\geq 3$$

We apply Cauchy's inequality: $\frac{y}{x}+\frac{x}{y}\geq 2\sqrt{\frac{y}{x}*\frac{x}{y}}$=2, $\frac{z}{x}+\frac{x}{z}\geq 2\sqrt{\frac{z}{x}*\frac{x}{z}}$=2, $\frac{x}{z}+\frac{z}{x}\geq 2\sqrt{\frac{x}{z}*\frac{z}{x}}$=2

$\frac{y}{x}+\frac{z}{x}+\frac{x}{y}+\frac{z}{y}+\frac{x}{z}+\frac{y}{z}\geq 6$

$\frac{y}{x}-2+\frac{z}{x}+\frac{x}{y}-2+\frac{z}{y}+\frac{x}{z}-2+\frac{y}{z}\geq 0$

$\frac{y}{x}-2\frac{\sqrt{y}}{\sqrt{x}}\frac{\sqrt{x}}{\sqrt{y}}+\frac{x}{y}+\frac{y}{z}-2\frac{\sqrt{y}}{\sqrt{z}}\frac{\sqrt{z}}{\sqrt{y}}+\frac{z}{y}+\frac{x}{z}-2\frac{\sqrt{x}}{\sqrt{z}}\frac{\sqrt{z}}{\sqrt{x}}+\frac{z}{x}\geq 0$

It is proved that: $(\frac{\sqrt{y}}{\sqrt{x}}-\frac{\sqrt{x}}{\sqrt{y}})^2+(\frac{\sqrt{y}}{\sqrt{z}}-\frac{\sqrt{z}}{\sqrt{y}})^2+(\frac{\sqrt{x}}{\sqrt{z}}-\frac{\sqrt{z}}{\sqrt{x}})^2\geq 0$

Example 5:

If $a^2+b^2+c^2+d^2=4$ prove the inequality $(a,b,c,d\in R)$ $a^3+b^3+c^3+d^3\geq 8$

Solution: : $a^2+b^2+c^2+d^2=4$ since $a^2\leq 4$ and $a\leq 2$,$a^2(a-2)\leq 0$ hence $a^3\leq 2a^2$, so $b^3\leq 2b^2, c^3\leq 2c^2$, $d^3\leq 2d^2$. Now we add the right side of the inequalities to the right and the left side to the left. It is proved that $a^3+b^3+c^3+d^3\geq 2(a^2+b^2+c^2+d^2)=8$

Example 6:

Prove the inequality

$$\frac{x}{y}+\frac{y}{z}+\frac{z}{x}\geq\frac{x+y+z}{\sqrt[3]{xyz}}$$

Solution: Let $\frac{x}{y}+\frac{y}{z}+\frac{z}{x}=A$ and multiply this expression by 3.

$3A=3*\left(\frac{x}{y}+\frac{y}{z}+\frac{z}{x}\right)=\left(\frac{2x}{y}+\frac{y}{z}\right)+\left(\frac{2y}{z}+\frac{z}{x}\right)+(\frac{2z}{x}+\frac{x}{y})$. Now we put the resulting expression into the Cauchy inequality.

It is proved: $\left(\frac{2x}{y}+\frac{y}{z}\right)+\left(\frac{2y}{z}+\frac{z}{x}\right)+\left(\frac{2z}{x}+\frac{x}{z}\right)\geq\frac{3x}{\sqrt[3]{xyz}}+\frac{3y}{\sqrt[3]{xyz}}+\frac{3z}{\sqrt[3]{xyz}}$

Example 7:

If a+b+c=1 for positive numbers a,b,c prove that the inequality holds $\frac{1}{1-a}+\frac{1}{1-b}+\frac{1}{1-c}\geq\frac{2}{1+a}+\frac{2}{1+b}+\frac{2}{1+c}$

Solution: $(a+b)\left(\frac{1}{a}+\frac{1}{b}\right)\geq 2\sqrt{ab}*2\frac{1}{\sqrt{ab}}=4$

$$\frac{1}{1-a}+\frac{1}{1-b}\geq\frac{4}{1-a+1-b}=\frac{4}{2-(a+b)}=\frac{4}{2-(1-c)}=\frac{4}{1+c}$$
$$\frac{1}{1-a}+\frac{1}{1-c}\geq\frac{4}{1-a+1-c}=\frac{4}{2-(a+c)}=\frac{4}{2-(1-b)}=\frac{4}{1+b}$$
$$\frac{1}{1-b}+\frac{1}{1-c}\geq\frac{4}{1-b+1-c}=\frac{4}{2-(b+c)}=\frac{4}{2-(1-a)}=\frac{4}{1+a}$$

Now we add the right side of the inequalities to the right and the left side to the left:

$$\frac{2}{1-a}+\frac{2}{1-b}+\frac{2}{1-c}\geq\frac{4}{1+a}+\frac{4}{1+b}+\frac{4}{1+c}$$

We divide this inequality by 2 and it is proved that:

$$\frac{1}{1-a}+\frac{1}{1-b}+\frac{1}{1-c}\geq\frac{2}{1+a}+\frac{2}{1+b}+\frac{2}{1+c}$$

Example 8: If $a^2+b^2+c^2=3$ for positive integers a,b,c, prove that the inequality $\frac{1}{1+ab}+\frac{1}{1+ac}+\frac{1}{1+bc}\geq\frac{3}{2}$

Solution: $a^2+b^2\geq 2ab$, $a^2+c^2\geq 2ac$, $b^2+c^2\geq 2bc$ from which

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$$3 = a^2 + c^2 + c^2 \geq ab + ac + bc$$

$$(a + b + c)\left(\frac{1}{a} + \frac{1}{b} + \frac{1}{c}\right) \geq 3\sqrt[3]{abc} * 3\sqrt[3]{\frac{1}{abc}} = 9$$

$$\left(\frac{1}{a} + \frac{1}{b} + \frac{1}{c}\right) \geq \frac{9}{a + b + c}$$

$$\frac{1}{1+ab} + \frac{1}{1+ac} + \frac{1}{1+bc} \geq \frac{9}{1+ab+1+ac+1+bc} = \frac{9}{3+ab+ac+bc} = \frac{9}{3+3}$$

$$it\ is\ proved\ that = \frac{3}{2}$$

Example 9:

If a, b, c > 0, abc = 1 then prove the following inequality:

$$\frac{1}{1 + a + b} + \frac{1}{1 + a + c} + \frac{1}{1 + d + c} \leq 1$$

Solution: We change the variables $= x^3$, $b = y^3$, $c = z^3$; $abc = x^3y^3z^3 = 1$ so $xyz = 1$

$$\frac{xyz}{xyz + x^3 + y^3} + \frac{xyz}{xyz + x^3 + z^3} + \frac{xyz}{xyz + y^3 + z^3} \leq 1$$

$(x - y)^2 \geq 0$

$x^2 - 2xy + y^2 \geq 0$

$x^2 - xy + y^2 \geq xy$

$x^2 - xy + y^2 \geq xy \quad /* (x + y)$

$(x + y)(x^2 - xy + y^2) \geq xy(x + y)$

$x^3 + y^3 \geq xy(x + y)$

So

$$\frac{xyz}{xyz+x^3+y^3} \leq \frac{xyz}{xyz+xy(x+y)} = \frac{z}{x+y+z}$$

$$\frac{xyz}{xyz + x^3 + z^3} \leq \frac{xyz}{xyz + xz(x + z)} = \frac{y}{x + y + z}$$

$$\frac{xyz}{xyz + y^3 + z^3} \leq \frac{xyz}{xyz + yz(x + z)} = \frac{x}{x + y + z}$$

Now we add the right side of the inequalities to the right and the left side to the left:

$$\frac{xyz}{xyz + x^3 + y^3} + \frac{xyz}{xyz + x^3 + z^3} + \frac{xyz}{xyz + y^3 + z^3} \leq \frac{z}{x + y + z} + \frac{y}{x + y + z} + \frac{x}{x + y + z}$$

$$\frac{xyz}{xyz + x^3 + y^3} + \frac{xyz}{xyz + x^3 + z^3} + \frac{xyz}{xyz + y^3 + z^3} \leq 1$$

Example 10:

Prove this inequality for positive real numbers

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$$\frac{a^3}{a^2+b^2}+\frac{b^3}{b^2+c^2}+\frac{c^3}{c^2+d^2}+\frac{d^3}{a^2+d^2}\geq\frac{a+b+c+d}{2}$$

$$\frac{a^3}{a^2+b^2}=a-\frac{ab^2}{a^{2+}b^2}\geq a-\frac{ab^2}{2ab}=a-\frac{b}{2}$$

$$\frac{b^3}{b^2+c^2}=b-\frac{bc^2}{b^{2+}c^2}\geq a-\frac{bc^2}{2bc}=b-\frac{c}{2}$$

$$\frac{c^3}{c^2+d^2}=c-\frac{cd^2}{c^{2+}d^2}\geq c-\frac{cd^2}{2cd}=c-\frac{d}{2}$$

$$\frac{d^3}{a^2+d^2}=d-\frac{da^2}{a^{2+}d^2}\geq d-\frac{da^2}{2ad}=d-\frac{a}{2}$$

Now we add the right side of the inequalities to the right and the left side to the left:

$$\frac{a^3}{a^2+b^2}+\frac{b^3}{b^2+c^2}+\frac{c^3}{c^2+d^2}+\frac{d^3}{a^2+d^2}\geq\ a-\frac{b}{2}+b-\frac{c}{2}+c-\frac{d}{2}+d-\frac{a}{2}$$

$\frac{a^3}{a^2+b^2}+\frac{b^3}{b^2+c^2}+\frac{c^3}{c^2+d^2}+\frac{d^3}{a^2+d^2}\geq\frac{a+b+c+d}{2}$ is proved.

CONCLUSION

In conclusion, every student who loves mathematics should learn how to solve these complex inequalities that are used in International Olympiads, as mentioned above, in order to increase their interest in this subject and develop their abilities. This will allow this student to lead their country to high achievements in International Olympiads, which means that science, technology and engineering will develop at a very high level in this country.

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