

**SOLVING A FINITE DIFFERENCE ONE-DIMENSIONAL PROBLEM USING THE
PROGONKA METHOD BASED ON AN IMPLICIT SCHEME**

<https://doi.org/10.5281/zenodo.18784637>

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Abstract: *In this article, a method for solving a one-dimensional heat conduction problem using an implicit finite difference scheme based on the integro-interpolation method and the sweep algorithm is considered. A non-uniform spatial difference grid and a uniform temporal grid are constructed, the differential equation is transformed into a discrete form, and as a result, a system of tridiagonal algebraic equations is obtained. Taking boundary conditions into account, an algorithm for determining the solution using the sweep (Thomas) method is presented. The forward and backward sweep stages are consistently analyzed, and the computational stability and efficiency of the method are substantiated. The proposed approach is of practical importance for accurate modeling of the temperature field in soil layers and for assessing the thermal regime of underground structures and foundation constructions.*

Keywords: *finite difference method, implicit difference scheme, one-dimensional problem, heat conduction equation, integro-interpolation method, sweep method, Thomas algorithm, tridiagonal system of equations, boundary conditions, difference grid, non-uniform step, stability, soil temperature, underground structures, mathematical modeling, numerical methods.*

INTRODUCTION

In the field of building physics and geotechnics, accurate modeling of heat exchange processes in soil layers is of great scientific and practical importance. During the design of underground structures, foundations, engineering communications, and energy facilities, the thermal regime of the soil directly affects the strength, stability, and long-term operation of structures. Therefore, solving the heat conduction equation using numerical methods is considered one of the most relevant problems.

Analytical methods usually have limited applicability in solving complex engineering problems, as they can be applied only to simplified geometries and boundary conditions. In practice, the heterogeneity of the physical and mechanical properties of soil layers, the

time-dependent variability of external influences, and the complexity of boundary conditions necessitate the use of numerical computational methods. One of these methods is the finite difference method, which makes it possible to solve differential equations by transforming them into a system of difference equations.

In this article, a one-dimensional heat conduction problem is discretized using an integro-interpolation approach based on an implicit finite difference scheme. To efficiently solve the resulting tridiagonal system of algebraic equations, the sweep (Thomas) method is applied. The proposed approach is analyzed in terms of computational stability, accuracy, and practical applicability.

Thus, the present study is aimed at developing an effective numerical method for determining the temperature field in soil and for evaluating the thermal regime of underground structures.

Main Part

Difference Grid (Discretization Method)

Solving differential equations by analytical methods is not always easy. Therefore, in practice, the problem is considered at discrete points, that is, instead of a continuous function, its values at certain points are taken. This method is called the difference grid method [1–5].

In this case:

- the grid along the coordinate (distance) is non-uniform (the steps may vary);
- the grid along time is uniform (the step is constant).

1. Spatial discretization.

Points along the spatial domain (for example, the x-axis) are defined as:

$$x_0, x_1, x_2, \dots, x_M$$

The spatial step is given by:

$$h_i = x_i - x_{i-1} \quad (1)$$

where h_i is the distance between the i -th point and the $(i-1)$ -th point.

The steps may be different:

$$h_1 \neq h_2 \neq h_3 \dots$$

This case is referred to as a non-uniform grid. This method is usually used to:

- increase accuracy near boundaries;
- better calculate regions where the gradient changes sharply.

2. Temporal discretization

Time points are defined as: $t_0, t_1, t_2, \dots, t_k$

The time step is:

$$\Delta t = t_k - t_{k-1} \quad (2)$$

where Δt is the constant time step.

All intervals are equal: $t_1 - t_0 = t_2 - t_1 = \dots = \Delta t$

This case is referred to as a uniform grid.

3. Indexing (numbering)

Spatial nodes: $i=0, 1, 2, \dots, M$

Time nodes: $k=0, 1, 2, \dots, K$

where M is the last node along the spatial coordinate (for example, the supporting boundary), and K is the final time step (the time at which the computation ends).

Why is this done? The continuous function $u(x,t)$ is now written in discrete form as:

$$u_i^k = u(x_i, t_k) \quad (3)$$

where i denotes the spatial location, and k denotes the time instant.

Thus, the continuous problem is transformed into a numerical problem in tabular form.

Implicit Difference Scheme

One of the most stable methods for solving the heat conduction equation by numerical methods is the implicit finite difference scheme [6–9].

In this method, the new values in time (the k layer) are unknown and are determined by solving a system of equations.

1. Discretization concept

Let the heat conduction equation in its general form be written as:

$$\frac{\partial t}{\partial \tau} = a \frac{\partial^2 t}{\partial x^2} \quad (4)$$

In the implicit scheme:

- the time derivative is approximated using the values at the previous and the new time levels;

- the second-order spatial derivative is calculated at the new time layer.

For this reason, the new time values become interdependent.

2. Resulting algebraic system

For each internal node ($i=1, 2, \dots, M-1$) an equation of the following form is obtained:

$$C_i t_i^k = A_i t_{i-1}^k + B_i t_{i+1}^k + F_i \quad (5)$$

This represents a tridiagonal system of algebraic equations.

Here: t_i^k is the temperature at the i -node at the k time layer;

t_{i-1}^k, t_{i+1}^k are the temperatures at the neighboring nodes;

A_i, B_i, C_i are coefficients;

F_i is the known term coming from the previous time layer.

That is, the temperature at each node is coupled with the temperatures of the neighboring nodes.

The coefficients depend on the following parameters:

- thermal conductivity coefficient λ ;
- density ρ ;
- specific heat capacity c ;
- spatial step h_i ;
- time step $\Delta\tau$.

Therefore, if the grid changes or the physical parameters change, the coefficients also change.

Why is it called “implicit”?

It is called “implicit” because:

- the unknowns at the new time layer appear simultaneously in the equation;
- they cannot be calculated directly;
- the system is solved (usually by the sweep method).

This scheme is stable, that is, it does not cause computational instability even for large time steps.

At each time step:

1. Boundary conditions are introduced;
2. A tridiagonal system is constructed;
3. The system is solved;
4. The temperatures at the new time layer are obtained.

Thus, the problem is solved step by step in time.

Taking boundary conditions into account

When solving a differential equation using the finite difference method, it is necessary to formulate separate equations not only for internal nodes but also for boundary nodes. This is because the solution of the problem depends on the physical conditions specified at the boundaries [10].

Let us assume that the problem is considered on the interval from $x=0$ до $x=H$.

1. Upper boundary ($x=0$)

The upper boundary is considered to be the soil surface. Here, usually:

- the temperature may be prescribed (Type I boundary condition);

• or heat exchange with the external environment is taken into account (Type III boundary condition).

Therefore, at the point $x=0$ the difference equation is written in a special form. That is, the formula used for internal nodes is not applied directly, but is adapted to the boundary condition.

2. Bottom Boundary ($x = H$)

At the bottom boundary, it is assumed that there is no heat flux. Physically, this means:

- there is no heat exchange at the bottom;
- or the temperature change with depth is very small.

Mathematical expression:

$$\frac{\partial t}{\partial x} = 0 \quad (6)$$

This is a second-type (Neumann) boundary condition.

3. Transformation to Difference Form

The result is written in a difference form. For example, at the last node ($i=M$):

$$\frac{\partial t}{\partial x} \approx \frac{t_M - t_{M-1}}{h_M} \quad (7)$$

According to the condition:

$$\frac{t_M - t_{M-1}}{h_M} = 0 \quad (8)$$

From this:

$$t_M = t_{M-1} \quad (9)$$

Thus, the temperature at the bottom boundary is equal to the temperature of the adjacent internal node.

4. Result

Thus:

- for internal nodes — the main difference equation,
- for the top boundary — a special equation,
- for the bottom boundary — $t_M = t_{M-1}$ the boundary condition is written.

These equations together form a general system of difference equations, and the solution of the problem is obtained through this system.

Algebraic system of equations

At each time step, a tridiagonal system of the following form is obtained [6–9]:

$$C_i t_i^k + A_i t_{i-1}^k + B_i t_{i+1}^k = F_i \quad (10)$$

To solve such systems, the most efficient method is the sweep method (Thomas algorithm).

Sweep Method (Thomas Algorithm)

The solution is sought in the following recursive form:

$$t_i^k = \tilde{\alpha}_{i+1} t_{i-1}^k + \tilde{\beta}_{i+1} \quad (11)$$

Here, $\tilde{\alpha}_i$ and $\tilde{\beta}_i$ are the **sweep coefficients**.

Starting from the boundary conditions, the coefficients $\tilde{\alpha}_i$ and $\tilde{\beta}_i$ are calculated sequentially. This process is called the **forward sweep**.

Then, the temperature at the bottom boundary is determined, and starting from this value, the temperatures at all nodes are computed **upwards**. This stage is called the **backward sweep**.

Scientific and Practical Significance

Scientific significance. The article theoretically justifies a method for solving the one-dimensional heat conduction problem using an explicit finite difference scheme. Through the integro-interpolation approach, a discrete analogue of the differential equation is obtained, and its stability and computational efficiency are analyzed. The application of the sweep method (Thomas algorithm) to solve the tridiagonal system of algebraic equations ensures an optimized numerical solution.

The research results provide a scientific basis for the advantages of explicit schemes in modeling heat transfer processes and contribute to the development of numerical methods theory used for determining the temperature field in soil layers.

Practical significance. The proposed method allows for an accurate assessment of the soil temperature regime in the design of underground structures, foundations, basements, underground utilities, and energy facilities. The simulation results can be applied to:

- evaluate the reliability of structures under freezing–thawing conditions;
- select appropriate thermal insulation parameters;
- calculate the energy efficiency of the underground environment;
- improve the accuracy of geotechnical and construction calculations;
- and solve other related engineering problems.

Thus, the developed numerical model serves as an effective tool for automating engineering calculations and optimizing the design process in practical applications.

CONCLUSION

Solving the one-dimensional heat conduction problem using an explicit finite difference scheme and the sweep (Thomas) method allows for accurate and stable calculation of the temperature field in soil. The explicit scheme ensures computational stability, while the sweep method enables fast and efficient solution of the tridiagonal system.

This approach is particularly important in the design of underground structures, foundations, and hydraulic engineering facilities. Correct selection of boundary conditions and optimal construction of the difference grid directly affects the accuracy of the computational results.

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