

ILL-POSED INITIAL–BOUNDARY VALUE PROBLEM FOR A SYSTEM OF MIXED-TYPE EQUATIONS WITH TWO DEGENERATE PLANES

<https://doi.org/10.5281/zenodo.21835995>

Y.K. Khudayberganov¹

J.I. Sharipov²

^{1,2}National University of Uzbekistan named after Mirzo Ulugbek

4, University St., Tashkent 700174, Uzbekistan

komilyashin89@mail.ru,

jasurbeksharipov55555@gmail.com

Abstract: This paper investigates an ill-posed initial–boundary value problem, in the sense of Hadamard, for a system of mixed-type equations with two degenerate planes. An appropriate correctness set is introduced for the considered problem, and sufficient conditions ensuring the uniqueness of the solution within this set are established. Furthermore, a conditional stability estimate for the solution is derived by means of appropriate a priori estimates. The obtained results contribute to the theory of ill-posed problems for systems of mixed-type equations with two degenerate planes and provide a theoretical foundation for the further investigation of boundary value problems.

Keywords: System of mixed-type equations, two degenerate planes, ill-posed initial–boundary value problem, correctness set, uniqueness of solution, conditional stability, a priori estimates, boundary value problem.

НЕКОРРЕКТНАЯ НАЧАЛЬНО-КРАЕВАЯ ЗАДАЧА ДЛЯ СИСТЕМЫ УРАВНЕНИЙ СМЕШАННОГО ТИПА С ДВУМЯ ПЛОСКОСТЯМИ ВЫРОЖДЕНИЯ

Я.К. Худайберганов¹

Ж.И. Шарипов²

^{1,2} Национальный университет Узбекистана имени Мирзо Улугбека

ул. Университетская, 4, Ташкент 100174, Узбекистан

komilyashin89@mail.ru

jasurbeksharipov55555@gmail.com

Аннотация. В настоящей работе исследуется некорректная начально-краевая задача в смысле Ж. Адамара для системы уравнений смешанного типа с двумя плоскостями вырождения. Для рассматриваемой задачи введено соответствующее множество корректности и установлены достаточные условия, обеспечивающие единственность решения в этом множестве. Кроме того, с помощью соответствующих априорных оценок получена условная оценка устойчивости решения. Полученные результаты развивают теорию некорректных задач для

систем уравнений смешанного типа с двумя плоскостями вырождения и создают теоретическую основу для дальнейшего исследования краевых задач.

Ключевые слова: Система уравнений смешанного типа, две плоскости вырождения, некорректная начально-краевая задача, множество корректности, единственность решения, условная устойчивость, априорные оценки, краевая задача.

INTRODUCTION

Mixed-type differential equations and their systems play an important role in modeling numerous applied problems arising in mathematical physics, gas dynamics, elasticity theory, heat transfer, filtration processes, and continuum mechanics. The study of the correctness, existence, uniqueness, and stability of solutions to initial-boundary value problems for such equations remains one of the most important research directions in the modern theory of partial differential equations.

The presence of degeneracy points or degeneracy planes in mixed-type equations significantly complicates their analysis. In particular, the vanishing of the equation coefficients on the degeneracy planes or the change in the equation type has a substantial influence on the qualitative properties of the solution. Therefore, establishing the existence of classical solutions, as well as investigating their uniqueness and stability, is of considerable theoretical importance.

According to Hadamard's definition, a problem is said to be ill-posed if at least one of the following conditions fails to hold: the existence of a solution, its uniqueness, or its continuous dependence on the given data. Significant contributions to the development of the theory of ill-posed problems have been made by A.N. Tikhonov[1], M.M. Lavrentiev[2], V.K. Ivanov[3], A.I. Kozhanov[4], S.G. Pyatkov[5],[6], S.A. Tersenov[7], K.S. Fayazov[8], A.A. Kerefov[9], I.E. Egorov[10], I.O. Khajiyev[11], Y.K. Khudayberganov[12] and many other researchers. In recent years, numerous results have been obtained concerning ill-posed initial-boundary value problems for parabolic and mixed-type equations with degeneracies. However, the study of such problems for systems of mixed-type equations with two degenerate planes has not yet been sufficiently developed.

In this paper, an ill-posed initial-boundary value problem, in the sense of Hadamard, is investigated for a system of mixed-type equations with two degenerate planes. An appropriate correctness set is introduced for the considered problem, and sufficient conditions ensuring the uniqueness of the solution within this set are established. Furthermore, conditional stability of the solution is derived by means of appropriate a priori estimates. The obtained results contribute to the theory of ill-posed problems for systems of mixed-type equations with two degenerate planes and provide a theoretical foundation for further investigations of this class of boundary value problems.

Given the domain $\Omega = \Omega_0 \times Q$, we consider the system of equations:

$$\begin{cases} u_{1tt} + \operatorname{sgn}(x)u_{1xx} + \operatorname{sgn}(y)u_{1yy} + a_1u_1 + b_1u_2 = 0, \\ u_{2tt} + \operatorname{sgn}(x)u_{2xx} + \operatorname{sgn}(y)u_{2yy} + a_2u_2 + b_2u_1 = 0, \end{cases} \quad (1)$$

where $\Omega_0 = \{(x, y) | (-1; 1) \times (-1; 1), x \neq 0, y \neq 0\}$, $Q = (0 < t < T, T < \infty)$,

a_1, a_2, b_1, b_2 – real numbers, $b_2 \neq 0$, $(a_1 - a_2)^2 + 4b_1b_2 > 0$.

Problem statement. We consider the problem of finding a pair of functions $(u_1(x, y, t), u_2(x, y, t))$ satisfying the system of equations (1) in the domain Ω and the following:

initial

$$\begin{aligned} \left. \frac{\partial^i u_1(x, y, t)}{\partial t^i} \right|_{t=0} &= \varphi_i(x, y), \quad (x, y) \in [-1; 1] \times [-1; 1], \\ \left. \frac{\partial^i u_2(x, y, t)}{\partial t^i} \right|_{t=0} &= \psi_i(x, y), \quad (x, y) \in [-1; 1] \times [-1; 1], i = 0, 1, \end{aligned} \quad (2)$$

boundary

$$\begin{aligned} u_j(x, y, t) \Big|_{\substack{x=-1 \\ x=+1}} &= 0, \quad (y, t) \in [-1; 1] \times \bar{Q}, \\ u_j(x, y, t) \Big|_{\substack{y=-1 \\ y=+1}} &= 0, \quad (x, t) \in [-1; 1] \times \bar{Q}, \quad j = 1, 2 \end{aligned} \quad (3)$$

and gluing

$$\begin{aligned} \left. \frac{\partial^i u_j(x, y, t)}{\partial x^i} \right|_{x=-0} &= \left. \frac{\partial^i u_j(x, y, t)}{\partial x^i} \right|_{x=+0}, \quad (y, t) \in [-1; 1] \times \bar{Q}, \\ \left. \frac{\partial^i u_j(x, y, t)}{\partial y^i} \right|_{y=-0} &= \left. \frac{\partial^i u_j(x, y, t)}{\partial y^i} \right|_{y=+0}, \quad (x, t) \in [-1; 1] \times \bar{Q}, \end{aligned} \quad (4)$$

conditions, where $(j = \overline{1, 2}), (i = \overline{0, 1})$, $\varphi_i(x, y), \psi_i(x, y)$ are given sufficiently smooth functions, furthermore, $\varphi_i(x, y)|_{\partial\Omega_0} = 0, \psi_i(x, y)|_{\partial\Omega_0} = 0$.

In solving the problem (1)-(4) we perform the following substitution

$$\begin{aligned} u_1(x, y, t) &= \frac{a_1 - \kappa_2}{b_2(\kappa_1 - \kappa_2)} v_1(x, y, t) - \frac{a_1 - \kappa_1}{b_2(\kappa_1 - \kappa_2)} v_2(x, y, t), \\ u_2(x, y, t) &= \frac{1}{\kappa_1 - \kappa_2} (v_1(x, y, t) - v_2(x, y, t)), \end{aligned} \quad (5)$$

where κ_1, κ_2 are the roots of this quadratic equation

$$\kappa^2 - (a_1 + a_2)\kappa + a_1a_2 - b_1b_2 = 0.$$

As a result, in finding the solution of the problem under consideration, with respect to the functions $v_1(x, y, t)$ and $v_2(x, y, t)$ we have the following:

Problem 1. The following

$$v_{1tt} + \operatorname{sgn}(x)v_{1xx} + \operatorname{sgn}(y)v_{1yy} + \kappa_1 v_1 = 0, \quad (6)$$

equation in the domain Ω as follows:

initial

$$\left. \frac{\partial^i v_1(x, y, t)}{\partial t^i} \right|_{t=0} = \bar{\varphi}_i(x, y), \quad (x, y) \in [-1; 1] \times [-1; 1], i = 0, 1 \quad (7)$$

boundary

$$\left. v_1(x, y, t) \right|_{\substack{x=-1 \\ x=+1}} = 0, \quad (y, t) \in [-1; 1] \times \bar{Q}, \quad (8)$$

$$\left. v_1(x, y, t) \right|_{\substack{y=-1 \\ y=+1}} = 0, \quad (x, t) \in [-1; 1] \times \bar{Q},$$

and gluing

$$\left. v_1(x, y, t) \right|_{\substack{x=-1 \\ x=+1}} = 0, \quad (y, t) \in [-1; 1] \times \bar{Q}, \quad (9)$$

$$\left. v_1(x, y, t) \right|_{\substack{y=-1 \\ y=+1}} = 0, \quad (x, t) \in [-1; 1] \times \bar{Q},$$

conditions, where $\bar{\varphi}_i(x, y) = b_2 \varphi_i(x, y) + (\kappa_1 - a_1) \psi_i(x, y)$, $i = 0, 1$.

Problem 2. The following

$$v_{2tt} + \operatorname{sgn}(x)v_{2xx} + \operatorname{sgn}(y)v_{2yy} + \kappa_2 v_2 = 0, \quad (10)$$

equation in the domain Ω as follows:

$$\left. \frac{\partial^i v_2(x, y, t)}{\partial t^i} \right|_{t=0} = \bar{\psi}_i(x, y), \quad (x, y) \in [-1; 1] \times [-1; 1], i = 0, 1 \quad (11)$$

initial,

$$\left. v_2(x, y, t) \right|_{\substack{x=-1 \\ x=+1}} = 0, \quad (y, t) \in [-1; 1] \times \bar{Q}, \quad (12)$$

$$\left. v_2(x, y, t) \right|_{\substack{y=-1 \\ y=+1}} = 0, \quad (x, t) \in [-1; 1] \times \bar{Q},$$

boundary,

$$\left. \frac{\partial^i v_2(x, y, t)}{\partial x^i} \right|_{x=-0} = \left. \frac{\partial^i v_2(x, y, t)}{\partial x^i} \right|_{x=+0}, \quad (y, t) \in [-1; 1] \times \bar{Q}, \quad (13)$$

$$\left. \frac{\partial^i v_2(x, y, t)}{\partial y^i} \right|_{y=-0} = \left. \frac{\partial^i v_2(x, y, t)}{\partial y^i} \right|_{y=+0}, \quad (x, t) \in [-1; 1] \times \bar{Q}, i = 0, 1$$

and satisfying the gluing conditions, leads to the problem of finding the functions $v_1(x, y, t)$ and $v_2(x, y, t)$, where $\bar{\psi}_i(x, y) = b_2 \varphi_i(x, y) + (\kappa_2 - a_1) \psi_i(x, y)$, $i = 0, 1$.

Spectral problem

Find the values of the parameter λ for which the following:

$$\operatorname{sgn}(x)\mathcal{G}_{xx}(x, y) + \operatorname{sgn}(y)\mathcal{G}_{yy}(x, y) + \lambda \mathcal{G}(x, y) = 0, \quad (x, y) \in \Omega_0, \quad (14)$$

$$\begin{aligned}
& \mathcal{G}(x, y) \Big|_{\substack{x=-1 \\ x=+1}} = 0, \quad y \in [-1; 1], \quad \mathcal{G}(x, y) \Big|_{\substack{y=-1 \\ y=+1}} = 0, \quad x \in [-1; 1], \\
& \frac{\partial^i \mathcal{G}(x, y)}{\partial x^i} \Big|_{x=-0} = \frac{\partial^i \mathcal{G}(x, y)}{\partial x^i} \Big|_{x=+0}, \quad y \in [-1; 1], \\
& \frac{\partial^i \mathcal{G}(x, y)}{\partial y^i} \Big|_{y=-0} = \frac{\partial^i \mathcal{G}(x, y)}{\partial y^i} \Big|_{y=+0}, \quad x \in [-1; 1], \quad (i = \overline{0, 1}),
\end{aligned} \tag{15}$$

problem has a non-trivial solution.

We use the Fourier method to solve the problem (14), (15). That is,

$$\mathcal{G}(x, y) = X(x) \cdot Y(y). \tag{16}$$

Using (16), we obtain the following conditions:

$$\begin{aligned}
X(-1) &= X(+1) = 0, \\
X(-0) &= X(+0), \\
X'(-0) &= X'(+0),
\end{aligned} \tag{17}$$

$$Y(-1) = Y(+1) = 0,$$

$$Y(-0) = Y(+0),$$

$$Y'(-0) = Y'(+0).$$

(18)

Calculating the second-order derivatives of (16):

$$\mathcal{G}_{xx}(x, y) = X''(x) \cdot Y(y),$$

$$\mathcal{G}_{yy}(x, y) = X(x) \cdot Y''(y), \quad \mathcal{G}_{xy}(x, y) = X(x) \cdot Y'(y).$$

Substituting into equation (14), we obtain the following:

$$\frac{\text{sgn}(x)X''(x)}{X(x)} + \frac{\text{sgn}(y)Y''(y)}{Y(y)} = -\lambda.$$

equality. Since the expression $\frac{\text{sgn}(x)X''(x)}{X(x)}$ in the equality above does not depend on

y , and the expression $\frac{\text{sgn}(y)Y''(y)}{Y(y)}$ does not depend on x , we have the following:

$$\frac{\text{sgn}(x)X''(x)}{X(x)} = -\lambda_1, \quad \frac{\text{sgn}(y)Y''(y)}{Y(y)} = -\lambda_2$$

equalities.

From this, for $X(x)$ and $Y(y)$ we have the following equations:

$$\text{sgn}(x)X''(x) = -\lambda_1 X(x), \tag{19}$$

$$\text{sgn}(y)Y''(y) = -\lambda_2 Y(y). \tag{20}$$

We consider equations (19) and (20) with conditions (17) and (18), respectively:

$$\operatorname{sgn}(x)X''(x) = -\lambda_1 X(x),$$

$$X(-1) = X(+1) = 0,$$

$$X(-0) = X(+0),$$

$$X'(-0) = X'(+0),$$

(21)

$$\operatorname{sgn}(y)Y''(y) = -\lambda_2 Y(y),$$

$$Y(-1) = Y(+1) = 0,$$

$$Y(-0) = Y(+0),$$

$$Y'(-0) = Y'(+0).$$

(22)

The solutions to the problems (21) and (22)

if $\lambda_1 > 0, \lambda_2 > 0$,

$$X_k^{(1)}(x) = \begin{cases} \sin \mu_k(x-1) / \cos \mu_k, & 0 \leq x \leq 1, \\ sh \mu_k(x+1) / ch \mu_k, & -1 \leq x \leq 0, \end{cases} \quad k \in N,$$

$$Y_l^{(1)}(y) = \begin{cases} \sin \sigma_l(y-1) / \cos \sigma_l, & 0 \leq y \leq 1, \\ sh \sigma_l(y+1) / ch \sigma_l, & -1 \leq y \leq 0, \end{cases} \quad l \in N,$$

if $\lambda_1 < 0, \lambda_2 < 0$,

$$X_k^{(2)}(x) = \begin{cases} sh \mu_k(x-1) / ch \mu_k, & 0 \leq x \leq 1, \\ \sin \mu_k(x+1) / \cos \mu_k, & -1 \leq x \leq 0, \end{cases} \quad k \in N,$$

$$Y_l^{(2)}(y) = \begin{cases} sh \sigma_l(y-1) / ch \sigma_l, & 0 \leq y \leq 1, \\ \sin \sigma_l(y+1) / \cos \sigma_l, & -1 \leq y \leq 0, \end{cases} \quad l \in N.$$

where $\lambda_{1k} = \mu_k^2 > 0, \lambda_{1k} = -\mu_k^2 < 0, \lambda_{2l} = \sigma_l^2 > 0, \lambda_{2l} = -\sigma_l^2 < 0$ are in the following form.

In both cases, μ_k, σ_l are the positive roots of the following $tg \alpha = -th \alpha$ the transcendental equation.

Thus, the eigenvalues of the spectral problem (14), (15) are $\mu_k^2 + \sigma_l^2 = \lambda_{k,l}^{(1)}, \mu_k^2 - \sigma_l^2 = \lambda_{k,l}^{(2)}, -\mu_k^2 + \sigma_l^2 = \lambda_{k,l}^{(3)}, -\mu_k^2 - \sigma_l^2 = \lambda_{k,l}^{(4)}$

in the form, and the corresponding eigenfunctions are

$$\mathcal{G}_{k,l}^{(1)}(x, y) = X_k^{(1)}(x) \cdot Y_l^{(1)}(y),$$

$$\mathcal{G}_{k,l}^{(2)}(x, y) = X_k^{(1)}(x) \cdot Y_l^{(2)}(y),$$

$$\mathcal{G}_{k,l}^{(3)}(x, y) = X_k^{(2)}(x) \cdot Y_l^{(1)}(y),$$

$$\mathcal{G}_{k,l}^{(4)}(x, y) = X_k^{(2)}(x) \cdot Y_l^{(2)}(y), \quad k, l \in N,$$

in the form.

According to the result of this [14] work, the eigenvalues of the spectral problem (14),

(15) $\{\lambda_{k,l}^{(1)}\}_{k,l=1}^{\infty}, \{-\lambda_{k,l}^{(2)}\}_{k,l=1}^{\infty}, \{\lambda_{k,l}^{(3)}\}_{k,l=1}^{\infty}, \{-\lambda_{k,l}^{(4)}\}_{k,l=1}^{\infty}$ form an increasing sequence and the eigenfunctions are $\{\mathcal{G}_{k,l}^{(j)}(x, y)\}_{k,l=1}^{\infty}, (j = \overline{1, 4})$.

Let $\|u\|^2 = (u, u)$, where $(u, v) = \int_{-1}^1 \int_{-1}^1 uv dx dy$, furthermore,

$$(sgn(x)sgn(y)\mathcal{G}_{k,l}^{(p)}(x, y), \mathcal{G}_{i,j}^{(q)}(x, y)) = 0, p \neq q, (p, q = \overline{1, 4}), \forall k, l, i, j,$$

$$(sgn(x)sgn(y)\mathcal{G}_{k,l}^{(m)}(x, y), \mathcal{G}_{i,j}^{(m)}(x, y)) = \begin{cases} 1, & k = i \wedge l = j, \\ 0, & k \neq i \wedge l \neq j, \end{cases} (m = 1, 4),$$

$$(sgn(x)sgn(y)\mathcal{G}_{k,l}^{(m)}(x, y), \mathcal{G}_{i,j}^{(m)}(x, y)) = \begin{cases} -1, & k = i \wedge l = j \\ 0, & k \neq i \wedge l \neq j \end{cases}, (m = 2, 3),$$

where $k, l, i, j \in N$.

The norm defined by the following formula:

$$\begin{aligned} \|u(x, y, t)\|_0^2 &= \sum_{k=1}^{\infty} \sum_{l=1}^{\infty} \left\{ \left| (sgn(x)sgn(y)u(x, y, t), \mathcal{G}_{k,l}^{(1)}(x, y)) \right|^2 + \right. \\ &+ \left| (sgn(x)sgn(y)u(x, y, t), \mathcal{G}_{k,l}^{(2)}(x, y)) \right|^2 + \\ &+ \left| (sgn(x)sgn(y)u(x, y, t), \mathcal{G}_{k,l}^{(3)}(x, y)) \right|^2 + \\ &\left. + \left| (sgn(x)sgn(y)u(x, y, t), \mathcal{G}_{k,l}^{(4)}(x, y)) \right|^2 \right\}, \end{aligned} \quad (23)$$

is equivalent to the initial norm in the space H_0 .

According to the results of theorem 2.1 in [16] and theorem 4.1 in [15], the eigenfunctions $\{\mathcal{G}_{k,l}^{(j)}(x, y)\}, (j = \overline{1, 4})$ of the spectral problem (14), (15) constitute a normalized Riesz basis in $L_2((-1; 1) \times (-1; 1))$

A priori estimate

By the generalized solution of the problem (6)-(9), we understand a function $v_1(x, y, t)$ such that $v_1(x, y, t), v_{1t}(x, y, t) \in C(L_2(-1, 1)^2, Q)$ and for any function $V(x, y, t) \in W_2^{2,2}((-1; 1), (-1; 1), (0; T))$, the following equality is satisfied:

$$\begin{aligned}
& \int_0^T \int_{-1}^1 \int_{-1}^1 v_1(x, y, t) \left(\operatorname{sgn}(x) \operatorname{sgn}(y) V_{tt}(x, y, t) + \operatorname{sgn}(y) V_{xx}(x, y, t) + \right. \\
& \quad \left. + \operatorname{sgn}(x) V_{yy}(x, y, t) + \operatorname{sgn}(x) \operatorname{sgn}(y) V(x, y, t) \right) dx dy dt = \\
& + \int_{-1}^1 \int_{-1}^1 \operatorname{sgn}(x) \operatorname{sgn}(y) V(x, y, 0) \bar{\varphi}_1(x, y) dx dy + \\
& - \int_{-1}^1 \int_{-1}^1 \operatorname{sgn}(x) \operatorname{sgn}(y) V_t(x, y, 0) \bar{\varphi}_0(x, y) dx dy,
\end{aligned}$$

where the conditions $\left. \frac{\partial^i V(x, y, t)}{\partial t^i} \right|_{t=T} = 0, i = 0, 1, V(-1, y, t) = V(1, y, t) = 0,$

$V(x, -1, t) = V(x, 1, t) = 0$ are fulfilled.

Let $M = \{u : \|u(x, y, z, T)\|_0 \leq m, m < \infty\}$.

Lemma 1. [17] Suppose the function $v(t)$ satisfies the equation $v''(t) - \eta v(t) = 0$ and the conditions $v(0) = p_1, v'(0) = p_2$. Then, for the function $v(t)$ and $t \in Q$, the following estimate holds:

$$v^2(t) \leq e^{2t(T-t)} \left(v^2(0) + |\alpha| \right)^{1-\frac{t}{T}} \left(v^2(T) + |\alpha| \right)^{\frac{t}{T}} - |\alpha|,$$

where η is an arbitrary number, and $\alpha = \frac{1}{2}(\eta v^2(0) - v_t^2(0))$.

Lemma 2. Suppose the function $v_1(x, y, t)$ satisfies equation (6) and conditions (7)-(9). Then for any $t \in Q$ the following estimate holds for the solution to the problem (6)-(9):

$$\|v_1(x, y, t)\|_0^2 \leq 4e^{2t(T-t)} \left(\|v_1(x, y, 0)\|_0^2 + \alpha_1 \right)^{1-\frac{t}{T}} \left(\|v_1(x, y, T)\|_0^2 + \alpha_1 \right)^{\frac{t}{T}} - \alpha_1, \quad (24)$$

where $\alpha_1 = \frac{1}{2}(\|\bar{\varphi}_0\|_1^2 + \|\bar{\varphi}_1\|_0^2)$.

Proof. If the solution to the problem (6)-(9) exists and $v_1(x, y, t) \in M$, then we can write the solution in the following form:

$$\begin{aligned}
v_1(x, y, t) = & \sum_{k=1}^{\infty} \sum_{l=1}^{\infty} \left(v_{1k,l}^{(1)}(t) \mathcal{G}_{k,l}^{(1)}(x, y) + v_{1k,l}^{(2)}(t) \mathcal{G}_{k,l}^{(2)}(x, y) + \right. \\
& \left. + v_{1k,l}^{(3)}(t) \mathcal{G}_{k,l}^{(3)}(x, y) + v_{1k,l}^{(4)}(t) \mathcal{G}_{k,l}^{(4)}(x, y) \right)
\end{aligned} \quad (25)$$

where $\{\mathcal{G}_{k,l}^{(j)}(x, y)\}_{k,l=1}^{\infty}, (j = \overline{1, 4})$ are the eigenfunctions of the (14)-(15) spectral problem. The functions are defined as follows:

$$\begin{aligned}
v_{1k,l}^{(j)}(t) &= (\operatorname{sgn}(x) \operatorname{sgn}(y) v_1(x, y, t), \mathcal{G}_{k,l}^{(j)}(x, y)), \quad (j = \overline{1, 4}), \\
v_{1k,l}^{(j)}(t) &= -(\operatorname{sgn}(x) \operatorname{sgn}(y) v_1(x, y, t), \mathcal{G}_{k,l}^{(j)}(x, y)), \quad (j = \overline{2, 3}),
\end{aligned}$$

$$\bar{\varphi}_{ik,l}^{(j)} = (\operatorname{sgn}(x)\operatorname{sgn}(y)\varphi_i(x, y), \mathcal{G}_{k,l}^{(j)}(x, y)), \quad (j = \overline{1, 4}),$$

$$\bar{\varphi}_{ik,l}^{(j)} = -(\operatorname{sgn}(x)\operatorname{sgn}(y)\varphi_i(x, y), \mathcal{G}_{k,l}^{(j)}(x, y)), \quad (j = \overline{2, 3}), \quad (i = \overline{0, 1}).$$

Let $V(x, y, t) = \omega_{k,l}(t)\mathcal{G}_{k,l}^{(j)}(x, y)$, $(j = \overline{1, 4})$, and let $\omega_{k,l}(T) = 0$, $\omega'_{k,l}(T) = 0$,

$\omega_{k,l}(t) \in W_2^2(Q)$. In this case, we have:

$$\int_0^T \int_{-1}^1 \int_{-1}^1 \operatorname{sgn} x \operatorname{sgn} y v_1(x, y, t) (\omega''_{k,l}(t) \mathcal{G}_{k,l}^{(j)}(x, y) - \omega(t) \lambda_{k,l}^{(j)} \mathcal{G}_{k,l}^{(j)}(x, y) + \kappa_2 \omega(t) \mathcal{G}_{k,l}^{(j)}(x, y)) dx dy dt =$$

$$\omega_{k,l}(0) \int_{-1}^1 \int_{-1}^1 \operatorname{sgn}(x) \operatorname{sgn}(y) \mathcal{G}_{k,l}^{(j)}(x, y) \varphi_1(x, y) dx dy - \omega'_{k,l}(0) \int_{-1}^1 \int_{-1}^1 \operatorname{sgn}(x) \operatorname{sgn}(y) \mathcal{G}_{k,l}^{(j)}(x, y) \varphi_0(x, y) dx dy.$$

According to the equality (26), it follows that:

$$\int_0^T v_{1k,l}^{(j)}(t) (\omega''_{k,l}(t) - (\lambda_{k,l}^{(j)} - \kappa_1) \omega_{k,l}(t)) dt = \omega_{k,l}(0) \varphi_{1k,l}^{(j)} - \omega'_{k,l}(0) \varphi_{0k,l}^{(j)}, \quad k, l \in N.$$

where

$$v_{1k,l}^{(j)}(t) = (\operatorname{sgn}(x)\operatorname{sgn}(y)v_1(x, y, t), \mathcal{G}_{k,l}^{(j)}(x, y)), \quad (j = \overline{1, 4}),$$

$$v_{1k,l}^{(j)}(t) = -(\operatorname{sgn}(x)\operatorname{sgn}(y)v_1(x, y, t), \mathcal{G}_{k,l}^{(j)}(x, y)), \quad (j = \overline{2, 3}),$$

$$\bar{\varphi}_{ik,l}^{(j)} = (\operatorname{sgn}(x)\operatorname{sgn}(y)\bar{\varphi}_i(x, y), \mathcal{G}_{k,l}^{(j)}(x, y)), \quad (j = \overline{1, 4}),$$

$$\bar{\varphi}_{ik,l}^{(j)} = -(\operatorname{sgn}(x)\operatorname{sgn}(y)\bar{\varphi}_i(x, y), \mathcal{G}_{k,l}^{(j)}(x, y)), \quad (j = \overline{2, 3}), \quad (i = \overline{0, 1}).$$

As a result, for $v_{1k,l}^{(j)}(t)$, $(j = \overline{1, 4})$, we arrive at the following problems:

$$(v_{1k,l}^{(j)}(t))_{tt} = (\lambda_{k,l}^{(j)} - \kappa_1) v_{1k,l}^{(j)}(t),$$

$$\left. \frac{\partial^i v_{1k,l}^{(j)}(t)}{\partial t^i} \right|_{t=0} = \bar{\varphi}_{ik,l}^{(j)}, \quad i = \overline{0, 1}, \quad j = \overline{1, 2, 3, 4}, \quad k, l \in N. \quad (27)$$

The solution to the problem (27) takes the form

$$v_{1k,l}^{(j)}(t) = \begin{cases} \bar{\varphi}_{0k,l}^{(j)} \operatorname{ch} \sqrt{\lambda_{k,l}^{(j)} - \kappa_1} t + \frac{\bar{\varphi}_{1k,l}^{(j)} \operatorname{sh} \sqrt{\lambda_{k,l}^{(j)} - \kappa_1} t}{\sqrt{\lambda_{k,l}^{(j)} - \kappa_1}}, & \lambda_{k,l}^{(j)} - \kappa_1 > 0, \quad (j = \overline{1, 3}), \\ \bar{\varphi}_{1k,l}^{(j)} t + \bar{\varphi}_{0k,l}^{(j)}, & \lambda_{k,l}^{(j)} - \kappa_1 = 0, \quad (j = \overline{2, 3}), \\ \bar{\varphi}_{0k,l}^{(j)} \cos \sqrt{|\lambda_{k,l}^{(j)} - \kappa_1|} t + \frac{\bar{\varphi}_{1k,l}^{(j)} \sin \sqrt{|\lambda_{k,l}^{(j)} - \kappa_1|} t}{\sqrt{|\lambda_{k,l}^{(j)} - \kappa_1|}}, & \lambda_{k,l}^{(j)} - \kappa_1 < 0, \quad (j = \overline{2, 4}), \quad k, l \in N. \end{cases}$$

(28)

From the equalities (25) and (28) we obtain the following equality:

$$\|v_1(x, y, t)\|_0^2 = \sum_{k=1}^{\infty} \sum_{l=1}^{\infty} \left(|v_{1k,l}^{(1)}(t)|^2 + |v_{1k,l}^{(2)}(t)|^2 + |v_{1k,l}^{(3)}(t)|^2 + |v_{1k,l}^{(4)}(t)|^2 \right), \quad (29)$$

We introduce the following norm:

$$\|\varphi_0(x, y)\|_1^2 = \sum_{k=1}^{\infty} \sum_{l=1}^{\infty} \left(|\lambda_{k,l}^{(1)}(\varphi_{0k,l}^{(1)})|^2 + |\lambda_{k,l}^{(2)}(\varphi_{0k,l}^{(2)})|^2 + |\lambda_{k,l}^{(3)}(\varphi_{0k,l}^{(3)})|^2 + |\lambda_{k,l}^{(4)}(\varphi_{0k,l}^{(4)})|^2 \right).$$

From the result of Lemma 1, for the solution of the problem (27), the following inequality holds for

$$\begin{aligned} (v_{1k,l}^{(j)}(t))^2 &\leq e^{2t(T-t)} \left((v_{1k,l}^{(j)}(0))^2 + \alpha_{1k,l}^{(j)} \right)^{1-\frac{t}{T}} \left((v_{1k,l}^{(j)}(T))^2 + \alpha_{1k,l}^{(j)} \right)^{\frac{t}{T}} - \\ &\quad - \alpha_{1k,l}^{(j)}, \quad t \in Q, \quad (j = \overline{1,4}), \end{aligned} \quad (30)$$

$$\text{where } \alpha_{1k,l}^{(j)} = \frac{1}{2} \left(\left((\lambda_{k,l}^{(j)} - \kappa_1) \right) (v_{1k,l}^{(j)}(0))^2 - (v_{1k,l}^{(j)}(0))^2 \right), \quad (j = \overline{1,4}). \quad (31)$$

We can write the equality (31) in the following form:

$$\alpha_{1k,l}^{(j)} \leq \frac{1}{2} \left(|\lambda_{k,l}^{(j)}| (v_{1k,l}^{(j)}(0))^2 + |\kappa_1| (v_{1k,l}^{(j)}(0))^2 + (v_{1k,l}^{(j)}(0))^2 \right), \quad (j = \overline{1,4}).$$

By summing the inequalities (30) for all $k, l, n \in N$ and applying the Hölder inequality, we arrive at the following result:

$$\begin{aligned} &\sum_{k=1}^{\infty} \sum_{l=1}^{\infty} \left((v_{1k,l}^{(1)}(t))^2 + (v_{1k,l}^{(2)}(t))^2 + (v_{1k,l}^{(3)}(t))^2 + (v_{1k,l}^{(4)}(t))^2 \right) \leq \\ &\leq 4e^{2t(T-t)} \left(\sum_{k=1}^{\infty} \sum_{l=1}^{\infty} \left((v_{1k,l}^{(1)}(0))^2 + (v_{1k,l}^{(2)}(0))^2 + (v_{1k,l}^{(3)}(0))^2 + (v_{1k,l}^{(4)}(0))^2 \right) + \right. \\ &\quad \left. + \sum_{k=1}^{\infty} \sum_{l=1}^{\infty} \left(\alpha_{k,l}^{(1)} + \alpha_{k,l}^{(2)} + \alpha_{k,l}^{(3)} + \alpha_{k,l}^{(4)} \right) \right)^{1-\frac{t}{T}} \times \\ &\times \left(\sum_{k=1}^{\infty} \sum_{l=1}^{\infty} \left((v_{1k,l}^{(1)}(T))^2 + (v_{1k,l}^{(2)}(T))^2 + (v_{1k,l}^{(3)}(T))^2 + (v_{1k,l}^{(4)}(T))^2 \right) + \right. \\ &\quad \left. + \sum_{k=1}^{\infty} \sum_{l=1}^{\infty} \left(\alpha_{k,l}^{(1)} + \alpha_{k,l}^{(2)} + \alpha_{k,l}^{(3)} + \alpha_{k,l}^{(4)} \right) \right)^{\frac{t}{T}} - \\ &\quad - \sum_{k=1}^{\infty} \sum_{l=1}^{\infty} \left(\alpha_{k,l}^{(1)} + \alpha_{k,l}^{(2)} + \alpha_{k,l}^{(3)} + \alpha_{k,l}^{(4)} \right), \end{aligned}$$

this leads to the following simplified inequality:

$$\|v_1(x, y, t)\|_0^2 \leq 4e^{2t(T-t)} \left(\|v_1(x, y, 0)\|_0^2 + \alpha_1 \right)^{1-\frac{t}{T}} \left(\|v_1(x, y, T)\|_0^2 + \alpha_1 \right)^{\frac{t}{T}} - \alpha_1,$$

where $\alpha_1 = \frac{1}{2} \left(\|\bar{\varphi}_0\|_1^2 + |\kappa_1| \|\bar{\varphi}_0\|_0^2 + \|\bar{\varphi}_1\|_0^2 \right).$

With this derivation, the proof of Lemma 2 is completed.

Uniqueness and Conditional Stability

Theorem 1. Suppose that the problem (1)–(4) has solutions $u_1(x, y, t), u_2(x, y, t) \in M$.

Then, for the solution of the problem (1)–(4), for any $t \in Q$, the following inequalities hold:

$$\begin{aligned} & \|u_1(x, y, t)\|_0^2 \leq \\ & 2 \left(\frac{a_1 - \kappa_2}{b_2(\kappa_1 - \kappa_2)} \right)^2 \left(4e^{2t(T-t)} \left(2b_2^2 \|\varphi_0(x, y)\|_0^2 + 2(\kappa_1 - a_1)^2 \|\psi_0(x, y)\|_0^2 + \alpha_1 \right)^{1-\frac{t}{T}} \times \right. \\ & \times \left(2b_2^2 \|u_1(x, y, T)\|_0^2 + 2(\kappa_1 - a_1)^2 \|u_2(x, y, T)\|_0^2 + \alpha_1 \right)^{\frac{t}{T}} - \alpha_1 \Big) + \\ & + 2 \left(\frac{a_1 - \kappa_1}{b_2(\kappa_1 - \kappa_2)} \right)^2 \left(4e^{2t(T-t)} \left(2b_2^2 \|\varphi_0(x, y)\|_0^2 + 2(\kappa_2 - a_1)^2 \|\psi_0(x, y)\|_0^2 + \alpha_2 \right)^{1-\frac{t}{T}} \times \right. \\ & \times \left(2b_2^2 \|u_1(x, y, T)\|_0^2 + 2(\kappa_2 - a_1)^2 \|u_2(x, y, T)\|_0^2 + \alpha_2 \right)^{\frac{t}{T}} - \alpha_2 \Big), \end{aligned}$$

$$\begin{aligned} & \|u_2(x, y, t)\|_0^2 \leq \\ & 2 \left(\frac{1}{\kappa_1 - \kappa_2} \right)^2 \left(4e^{2t(T-t)} \left(2b_2^2 \|\varphi_0(x, y)\|_0^2 + 2(\kappa_1 - a_1)^2 \|\psi_0(x, y)\|_0^2 + \alpha_1 \right)^{1-\frac{t}{T}} \times \right. \\ & \times \left(2b_2^2 \|u_1(x, y, T)\|_0^2 + 2(\kappa_1 - a_1)^2 \|u_2(x, y, T)\|_0^2 + \alpha_1 \right)^{\frac{t}{T}} - \alpha_1 + \\ & 4e^{2t(T-t)} \left(2b_2^2 \|\varphi_0(x, y)\|_0^2 + 2(\kappa_2 - a_1)^2 \|\psi_0(x, y)\|_0^2 + \alpha_2 \right)^{1-\frac{t}{T}} \times \\ & \times \left(2b_2^2 \|u_1(x, y, T)\|_0^2 + 2(\kappa_2 - a_1)^2 \|u_2(x, y, T)\|_0^2 + \alpha_2 \right)^{\frac{t}{T}} - \alpha_2 \Big) \end{aligned}$$

where

$$\begin{aligned} \alpha_1 &= \frac{1}{2} \left(\|\bar{\varphi}_0\|_1^2 + |\kappa_1| \|\bar{\varphi}_0\|_0^2 + \|\bar{\varphi}_1\|_0^2 \right), \\ \alpha_2 &= \frac{1}{2} \left(\|\bar{\psi}_0\|_1^2 + |\kappa_2| \|\bar{\psi}_0\|_0^2 + \|\bar{\psi}_1\|_0^2 \right). \end{aligned}$$

Proof. By Lemma 2, for the solutions of the problems (6)–(9) and (10)–(13), the following inequalities hold:

$$\begin{aligned} \|v_1(x, y, t)\|_0^2 &\leq 4e^{2t(T-t)} \left(\|v_1(x, y, 0)\|_0^2 + \alpha_1 \right)^{1-\frac{t}{T}} \left(\|v_1(x, y, T)\|_0^2 + \alpha_1 \right)^{\frac{t}{T}} - \alpha_1, \\ \|v_2(x, y, t)\|_0^2 &\leq 4e^{2t(T-t)} \left(\|v_2(x, y, 0)\|_0^2 + \alpha_2 \right)^{1-\frac{t}{T}} \left(\|v_2(x, y, T)\|_0^2 + \alpha_2 \right)^{\frac{t}{T}} - \alpha_2, \quad (32) \end{aligned}$$

where

$$\alpha_1 = \frac{1}{2} \left(\|\bar{\varphi}_0\|_1^2 + |\kappa_1| \|\bar{\varphi}_0\|_0^2 + \|\bar{\varphi}_1\|_0^2 \right), \quad \alpha_2 = \frac{1}{2} \left(\|\bar{\psi}_0\|_1^2 + |\kappa_2| \|\bar{\psi}_0\|_0^2 + \|\bar{\psi}_1\|_0^2 \right).$$

Using (5), we obtain:

$$\begin{aligned} v_1(x, y, t) &= b_2 u_1(x, y, t) + (\kappa_1 - a_1) u_2(x, y, t), \\ v_2(x, y, t) &= b_2 u_1(x, y, t) + (\kappa_2 - a_1) u_2(x, y, t), \end{aligned} \quad (33)$$

$$\begin{aligned} \|u_1(x, y, t)\|_0 &\leq \left| \frac{a_1 - \kappa_2}{b_2(\kappa_1 - \kappa_2)} \right| \|v_1(x, y, t)\|_0 + \left| \frac{a_1 - \kappa_1}{b_2(\kappa_1 - \kappa_2)} \right| \|v_2(x, y, t)\|_0, \\ \|u_2(x, y, t)\|_0 &\leq \left| \frac{1}{\kappa_1 - \kappa_2} \right| \left(\|v_1(x, y, t)\|_0 + \|v_2(x, y, t)\|_0 \right). \end{aligned} \quad (34)$$

Based on inequalities (24), (32), (33), and (34), we obtain the following estimates for the norms of the functions $u_1(x, y, t)$ and $u_2(x, y, t)$:

$$\begin{aligned} \|u_1(x, y, t)\|_0^2 &\leq \\ &2 \left(\frac{a_1 - \kappa_2}{b_2(\kappa_1 - \kappa_2)} \right)^2 \left(4e^{2t(T-t)} \left(2b_2^2 \|\varphi_0(x, y)\|_0^2 + 2(\kappa_1 - a_1)^2 \|\psi_0(x, y)\|_0^2 + \alpha_1 \right)^{1-\frac{t}{T}} \times \right. \\ &\times \left. \left(2b_2^2 \|u_1(x, y, T)\|_0^2 + 2(\kappa_1 - a_1)^2 \|u_2(x, y, T)\|_0^2 + \alpha_1 \right)^{\frac{t}{T}} - \alpha_1 \right) + \\ &+ 2 \left(\frac{a_1 - \kappa_1}{b_2(\kappa_1 - \kappa_2)} \right)^2 \left(4e^{2t(T-t)} \left(2b_2^2 \|\varphi_0(x, y)\|_0^2 + 2(\kappa_2 - a_1)^2 \|\psi_0(x, y)\|_0^2 + \alpha_2 \right)^{1-\frac{t}{T}} \times \right. \\ &\times \left. \left(2b_2^2 \|u_1(x, y, T)\|_0^2 + 2(\kappa_2 - a_1)^2 \|u_2(x, y, T)\|_0^2 + \alpha_2 \right)^{\frac{t}{T}} - \alpha_2 \right), \end{aligned} \quad (35)$$

$$\begin{aligned} \|u_2(x, y, t)\|_0^2 &\leq \\ &2 \left(\frac{1}{\kappa_1 - \kappa_2} \right)^2 \left(4e^{2t(T-t)} \left(2b_2^2 \|\varphi_0(x, y)\|_0^2 + 2(\kappa_1 - a_1)^2 \|\psi_0(x, y)\|_0^2 + \alpha_1 \right)^{1-\frac{t}{T}} \times \right. \\ &\times \left. \left(2b_2^2 \|u_1(x, y, T)\|_0^2 + 2(\kappa_1 - a_1)^2 \|u_2(x, y, T)\|_0^2 + \alpha_1 \right)^{\frac{t}{T}} - \alpha_1 + \right. \\ &4e^{2t(T-t)} \left(2b_2^2 \|\varphi_0(x, y)\|_0^2 + 2(\kappa_2 - a_1)^2 \|\psi_0(x, y)\|_0^2 + \alpha_2 \right)^{1-\frac{t}{T}} \times \\ &\times \left. \left(2b_2^2 \|u_1(x, y, T)\|_0^2 + 2(\kappa_2 - a_1)^2 \|u_2(x, y, T)\|_0^2 + \alpha_2 \right)^{\frac{t}{T}} - \alpha_2 \right) \end{aligned} \quad (36)$$

This completes the proof of Theorem 1.

Theorem 2. If a solution to the problem (1)–(4) exists and $(u_1(x, y, t), u_2(x, y, t)) \in M$, then the solution to the problem (1)–(4) is unique.

Proof. Assume that the pair of functions $(u_{1,1}(x, y, t), u_{2,1}(x, y, t))$, $(u_{1,2}(x, y, t), u_{2,2}(x, y, t))$ are solutions to the problem (1)–(4). We introduce the following

notation: $U_1(x, y, t) = u_{1,1}(x, y, t) - u_{1,2}(x, y, t)$ and $U_2(x, y, t) = u_{2,1}(x, y, t) - u_{2,2}(x, y, t)$

. Then, the pair of functions $(U_1(x, y, t), U_2(x, y, t))$ satisfies the equation:

$$\begin{cases} U_{1tt} + \operatorname{sgn}(x)U_{1xx} + \operatorname{sgn}(y)U_{1yy} + a_1U_1 + b_1U_2 = 0, \\ U_{2tt} + \operatorname{sgn}(x)U_{2xx} + \operatorname{sgn}(y)U_{2yy} + a_2U_2 + b_2U_1 = 0, \end{cases} \quad (37)$$

and the initial conditions:

$$\begin{aligned} \left. \frac{\partial^i U_1(x, y, t)}{\partial t^i} \right|_{t=0} &= 0, \quad (x, y) \in [-1; 1]^2, \\ \left. \frac{\partial^i U_2(x, y, t)}{\partial t^i} \right|_{t=0} &= 0, \quad (x, y) \in [-1; 1]^2, i = 0, 1, \end{aligned} \quad (38)$$

as well as the boundary conditions:

$$\begin{aligned} U_j(x, y, t) \Big|_{\substack{x=-1 \\ x=+1}} &= 0, \quad (y, t) \in [-1; 1] \times \bar{Q}, \\ U_j(x, y, t) \Big|_{\substack{y=-1 \\ y=+1}} &= 0, \quad (x, t) \in [-1; 1] \times \bar{Q}, \quad j = 1, 2 \end{aligned} \quad (39)$$

Applying the inequalities (34) and (35) to the solution of the problem (37)–(39), we obtain:

$$\|U_1(x, y, t)\| \leq 0,$$

$$\|U_2(x, y, t)\| \leq 0$$

It follows that:

$$\|U_1(x, y, t)\| = 0,$$

$$\|U_2(x, y, t)\| = 0.$$

This demonstrates that $u_{1,1}(x, y, t) \equiv u_{1,2}(x, y, t)$, $u_{2,1}(x, y, t) \equiv u_{2,2}(x, y, t)$ for $\forall (x, y, t) \in \Omega$. This completes the proof of Theorem 2.

Assume that the pair of functions $(u_1(x, y, t), u_2(x, y, t))$ is the solution to the problem (1)–(4) with exact data, while the pair of functions $(u_{1\varepsilon}(x, y, t), u_{2\varepsilon}(x, y, t))$ is the solution to the problem (1)–(4) with approximate data.

Theorem 3. Suppose that $(u_1(x, y, t), u_2(x, y, t)) \in M$, $(u_{1\varepsilon}(x, y, t), u_{2\varepsilon}(x, y, t)) \in M$

and $\|\varphi_i(x, y) - \varphi_{i\varepsilon}(x, y)\|_1 \leq \varepsilon$, $\|\psi_i(x, y) - \psi_{i\varepsilon}(x, y)\|_1 \leq \varepsilon$, $i = 0, 1$. Then, for the functions $U_{1\varepsilon}(x, y, t)$, $U_{2\varepsilon}(x, y, t)$, the following estimates hold for $t \in (0; T)$

$$\begin{aligned} & \|U_{1\varepsilon}(x, y, t)\|_0^2 \leq \\ & 2 \left(\frac{a_1 - \kappa_2}{b_2(\kappa_1 - \kappa_2)} \right)^2 \left(4e^{2t(T-t)} \left(2b_2^2 \varepsilon^2 + 2(\kappa_1 - a_1)^2 \varepsilon^2 + (2 + |\kappa_1|) \varepsilon^2 (|b|^2 + |\kappa_1 - a_1|^2) \right)^{1-\frac{t}{T}} \times \right. \\ & \times \left(8b_2^2 m^2 + 8(\kappa_1 - a_1)^2 m^2 + (2 + |\kappa_1|) \varepsilon^2 (|b|^2 + |\kappa_1 - a_1|^2) \right)^{\frac{t}{T}} - (2 + |\kappa_1|) \varepsilon^2 (|b|^2 + |\kappa_1 - a_1|^2) \Bigg) + \\ & + 2 \left(\frac{a_1 - \kappa_1}{b_2(\kappa_1 - \kappa_2)} \right)^2 \left(4e^{2t(T-t)} \left(8b_2^2 \varepsilon^2 + 8(\kappa_2 - a_1)^2 \varepsilon^2 + (2 + |\kappa_2|) \varepsilon^2 (|b|^2 + |\kappa_2 - a_1|^2) \right)^{1-\frac{t}{T}} \times \right. \\ & \times \left(8b_2^2 m^2 + 8(\kappa_2 - a_1)^2 m^2 + (2 + |\kappa_2|) \varepsilon^2 (|b|^2 + |\kappa_2 - a_1|^2) \right)^{\frac{t}{T}} - (2 + |\kappa_2|) \varepsilon^2 (|b|^2 + |\kappa_2 - a_1|^2) \Bigg), \end{aligned}$$

$$\begin{aligned} & \|U_{2\varepsilon}(x, y, t)\|_0^2 \leq \\ & 2 \left(\frac{1}{\kappa_1 - \kappa_2} \right)^2 \left(4e^{2t(T-t)} \left(2b_2^2 \varepsilon^2 + 2(\kappa_1 - a_1)^2 \varepsilon^2 + (2 + |\kappa_1|) \varepsilon^2 (|b|^2 + |\kappa_1 - a_1|^2) \right)^{1-\frac{t}{T}} \times \right. \\ & \times \left(8b_2^2 m^2 + 8(\kappa_1 - a_1)^2 m^2 + (2 + |\kappa_1|) \varepsilon^2 (|b|^2 + |\kappa_1 - a_1|^2) \right)^{\frac{t}{T}} - (2 + |\kappa_1|) \varepsilon^2 (|b|^2 + |\kappa_1 - a_1|^2) + \\ & 4e^{2t(T-t)} \left(2b_2^2 \varepsilon^2 + 2(\kappa_2 - a_1)^2 \varepsilon^2 + (2 + |\kappa_2|) \varepsilon^2 (|b|^2 + |\kappa_2 - a_1|^2) \right)^{1-\frac{t}{T}} \times \\ & \times \left(8b_2^2 m^2 + 8(\kappa_2 - a_1)^2 m^2 + (2 + |\kappa_2|) \varepsilon^2 (|b|^2 + |\kappa_2 - a_1|^2) \right)^{\frac{t}{T}} - (2 + |\kappa_2|) \varepsilon^2 (|b|^2 + |\kappa_2 - a_1|^2) \Bigg) \end{aligned}$$

Proof. We introduce the following notation: $U_{1\varepsilon}(x, y, t) = u_1(x, y, t) - u_{1\varepsilon}(x, y, t)$, $U_{2\varepsilon}(x, y, t) = u_2(x, y, t) - u_{2\varepsilon}(x, y, t)$. Then, for the pair of functions $(U_{1\varepsilon}(x, y, t), U_{2\varepsilon}(x, y, t))$, we have the following system of equations:

$$\begin{cases} U_{1\varepsilon tt} + \operatorname{sgn}(x)U_{1\varepsilon xx} + \operatorname{sgn}(y)U_{1\varepsilon yy} + a_1 U_{1\varepsilon} + b_1 U_{2\varepsilon} = 0, \\ U_{2\varepsilon tt} + \operatorname{sgn}(x)U_{2\varepsilon xx} + \operatorname{sgn}(y)U_{2\varepsilon yy} + a_2 U_{2\varepsilon} + b_2 U_{1\varepsilon} = 0, \end{cases} \quad (40)$$

and the initial conditions take the form:

$$\begin{aligned} \frac{\partial^i U_{1\varepsilon}(x, y, t)}{\partial t^i} &= \bar{\varphi}_i(x, y) - \bar{\varphi}_{i\varepsilon}(x, y), \\ \frac{\partial^i U_{2\varepsilon}(x, y, t)}{\partial t^i} &= \bar{\psi}_i(x, y) - \bar{\psi}_{i\varepsilon}(x, y), i = 0, 1. \end{aligned} \quad (41)$$

According to the conditions of the theorem, considering the problem (40), (41) where $\|\varphi_i(x, y) - \varphi_{i\varepsilon}(x, y)\|_1 \leq \varepsilon$, $\|\psi_i(x, y) - \psi_{i\varepsilon}(x, y)\|_1 \leq \varepsilon$, $i = 0, 1$, $(u_1(x, y, t), u_2(x, y, t)) \in M$, $(u_{1\varepsilon}(x, y, t), u_{2\varepsilon}(x, y, t)) \in M$, we perform the following estimations:

$$\begin{aligned} \|\bar{\varphi}_i(x, y) - \bar{\varphi}_{i\varepsilon}(x, y)\|_0 &\leq |b| \|\varphi_i(x, y) - \varphi_{i\varepsilon}(x, y)\|_0 + |\kappa_1 - a_1| \|\psi_i(x, y) - \psi_{i\varepsilon}(x, y)\|_0 \leq \\ &\leq \varepsilon (|b| + |\kappa_1 - a_1|), i = 0, 1, \end{aligned}$$

$$\begin{aligned} \|\bar{\psi}_i(x, y) - \bar{\psi}_{i\varepsilon}(x, y)\|_0 &\leq |b| \|\varphi_i(x, y) - \varphi_{i\varepsilon}(x, y)\|_0 + |\kappa_2 - a_1| \|\psi_i(x, y) - \psi_{i\varepsilon}(x, y)\|_0 \leq \\ &\leq \varepsilon (|b| + |\kappa_2 - a_1|), i = 0, 1, \end{aligned}$$

$$\alpha_1 \leq (2 + |\kappa_1|) \varepsilon^2 (|b|^2 + |\kappa_1 - a_1|^2),$$

$$\alpha_2 \leq (2 + |\kappa_2|) \varepsilon^2 (|b|^2 + |\kappa_2 - a_1|^2),$$

$$\|U_{1\varepsilon}(x, y, T)\|_0 \leq$$

$$\|u_1(x, y, T) - u_{1\varepsilon}(x, y, T)\|_0^2 \leq 2 \left(\|u_1(x, y, T)\|_0^2 + \|u_{1\varepsilon}(x, y, T)\|_0^2 \right) \leq 4m^2,$$

$$\|U_{2\varepsilon}(x, y, T)\|_0 \leq$$

$$\|u_2(x, y, T) - u_{2\varepsilon}(x, y, T)\|_0^2 \leq 2 \left(\|u_2(x, y, T)\|_0^2 + \|u_{2\varepsilon}(x, y, T)\|_0^2 \right) \leq 4m^2.$$

Using the aforementioned estimates and the result of Theorem 1, we obtain the following estimates:

$$\begin{aligned} \|U_{1\varepsilon}(x, y, t)\|_0^2 &\leq \\ &2 \left(\frac{a_1 - \kappa_2}{b_2(\kappa_1 - \kappa_2)} \right)^2 \left(4e^{2t(T-t)} \left(2b_2^2 \varepsilon^2 + 2(\kappa_1 - a_1)^2 \varepsilon^2 + (2 + |\kappa_1|) \varepsilon^2 (|b|^2 + |\kappa_1 - a_1|^2) \right)^{1-\frac{t}{T}} \times \right. \\ &\times \left(8b_2^2 m^2 + 8(\kappa_1 - a_1)^2 m^2 + (2 + |\kappa_1|) \varepsilon^2 (|b|^2 + |\kappa_1 - a_1|^2) \right)^{\frac{t}{T}} - (2 + |\kappa_1|) \varepsilon^2 (|b|^2 + |\kappa_1 - a_1|^2) \Big) + \\ &+ 2 \left(\frac{a_1 - \kappa_1}{b_2(\kappa_1 - \kappa_2)} \right)^2 \left(4e^{2t(T-t)} \left(8b_2^2 \varepsilon^2 + 8(\kappa_2 - a_1)^2 \varepsilon^2 + (2 + |\kappa_2|) \varepsilon^2 (|b|^2 + |\kappa_2 - a_1|^2) \right)^{1-\frac{t}{T}} \times \right. \\ &\times \left(8b_2^2 m^2 + 8(\kappa_2 - a_1)^2 m^2 + (2 + |\kappa_2|) \varepsilon^2 (|b|^2 + |\kappa_2 - a_1|^2) \right)^{\frac{t}{T}} - (2 + |\kappa_2|) \varepsilon^2 (|b|^2 + |\kappa_2 - a_1|^2) \Big), \\ \|U_{2\varepsilon}(x, y, t)\|_0^2 &\leq \\ &2 \left(\frac{1}{\kappa_1 - \kappa_2} \right)^2 \left(4e^{2t(T-t)} \left(2b_2^2 \varepsilon^2 + 2(\kappa_1 - a_1)^2 \varepsilon^2 + (2 + |\kappa_1|) \varepsilon^2 (|b|^2 + |\kappa_1 - a_1|^2) \right)^{1-\frac{t}{T}} \times \right. \\ &\times \left(8b_2^2 m^2 + 8(\kappa_1 - a_1)^2 m^2 + (2 + |\kappa_1|) \varepsilon^2 (|b|^2 + |\kappa_1 - a_1|^2) \right)^{\frac{t}{T}} - (2 + |\kappa_1|) \varepsilon^2 (|b|^2 + |\kappa_1 - a_1|^2) + \\ &4e^{2t(T-t)} \left(2b_2^2 \varepsilon^2 + 2(\kappa_2 - a_1)^2 \varepsilon^2 + (2 + |\kappa_2|) \varepsilon^2 (|b|^2 + |\kappa_2 - a_1|^2) \right)^{1-\frac{t}{T}} \times \\ &\times \left(8b_2^2 m^2 + 8(\kappa_2 - a_1)^2 m^2 + (2 + |\kappa_2|) \varepsilon^2 (|b|^2 + |\kappa_2 - a_1|^2) \right)^{\frac{t}{T}} - (2 + |\kappa_2|) \varepsilon^2 (|b|^2 + |\kappa_2 - a_1|^2) \Big) \end{aligned}$$

Theorem 3 is proved.

REFERENCES:

1. Tikhonov, A. N., Arsenin, V. Y. *Solutions of Ill-Posed Problems*. Winston & Sons, Washington, D.C., 1977.
2. Lavrentiev, M. M., Romanov, V. G., Shishatskii, S. P. *Ill-Posed Problems of Mathematical Physics and Analysis*. American Mathematical Society, Providence, RI, 1986.
3. Ivanov, V. K., Vasin, V. V., Tanana, V. P. *Theory of Linear Ill-Posed Problems and Its Applications*. VSP, Utrecht, 2002.
4. Kozhanov A. I. Well-posedness and ill-posedness of boundary-value problems for one class of fourth-order differential equations of Sobolev type // *Itogi Nauki i Tekhniki. Sovremennaya Matematika i ee Prilozheniya. Tematicheskie Obzory*. 2021. Vol. 198. P. 68–75
5. Pyatkov, S. G. *Some Classes of Ill-Posed Problems*. VSP, Utrecht, 2012.
6. Pyatkov, S. G. *Operator Theory. Nonclassical Problems*. VSP, Utrecht, 2002.
7. Tersenov, S. A. *Parabolic Equations with Changing Time Direction*. Novosibirsk: Nauka, 1985. (*rus tilida*)
8. Fayazov K. S., Khudaybergenov Ya. K. An ill-posed boundary value problem for a system of mixed-type equations with two degenerate planes, *Siberian Electronic Mathematical Reports*, Vol. 17, pp. 647–660 (2020). (in Russian)
9. Kerefov, A. A. *Mixed-Type Equations and Boundary Value Problems*. Vladikavkaz, 2000.
10. Egorov, I. E. "Boundary value problems for parabolic equations with degeneration." *Differential Equations*, 35 (1999), 1123–1131.
11. Khazhiev I. O. Conditional stability of the boundary value problem for a high-order system of mixed-type equations, *Scientific Bulletin of SamSU*, No. 1, pp. 60–67 (2022). (in Russian)
12. Fayazov, K.S., Khudayberganov, Y.K. Nonlocal boundary value problem for a nonhomogeneous parabolic type equation with two degenerate planes, *Uzbek Mathematical Journal Open source preview*, 2024, 68(3), Pp 53–65.
13. Fayazov, K.S., Khudayberganov, Y.K., Pyatkov S.G. *Journal of Mathematical Sciences United States Open source preview*, 2023, 274(2), Pp 201–214.
14. Пятков С.Г., Неклассические дифференциальные уравнения в частных производных // Новосибирск, (1988), 77 с.
15. Пятков С.Г., Абашеева Н.Л., Разрешимость краевых задач для операторно-дифференциальных уравнений смешанного типа. Вырожденный случай, *Сиб. матем. журн.*, 43(3) (2002), с. 678–693.
16. Крейн М.Г., Гохберг И. Ц. Введение в теорию линейных несамосопряженных операторов в гильбертовом пространстве // М.: Наука (1965). 448 с.
17. Лаврентьев М. М., Савельев Л. Я., Теория операторов и некорректные задачи // 2-е изд. Новосибирск: Изд-во Ин-та математики. (2010), 941 с.