ON A MATHEMATICAL MODELING OF A DAMPED MOTION

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Abstract:*In this paper, we construct a mathematical model of a damped using two different wave equation. The process is linked with vibration of stretched string taking place in a special medium, where damping force acting in a short time and then disappears for the rest of the time. The key tool is a method of a separation of variables.*

Keywords: *wave equation, damping force, method of separation variables.*

It is well-known that motion caused by vibration of string due to stretch can be modeled by wave equation $[1]$. If it happens in a special media, where damping force is acting then model wave equation will be modified adding $\frac{\partial u}{\partial t}$ term. There are many works devoted to the studying spring-mass-damping processes, related with integer or fractional order differential equations [2-4].

Let us consider the following problem:

<u>Problem.</u> A string is stretched and secured on the x-axis at $x=0$=0 and $x=\pi$ for some time-interval, for instance, $t\in(0,T)$, where $T > 0$ is a real number.

If the transverse vibrations take place in a medium that imports a resistance proportional to the instantaneous velocity depending on time and the initial displacement is $\varphi(\mathrm{x})$ find the equation of the motion.

A mathematical model of this process is based on wave equation of the following form

$$\frac{\partial^2 u}{\partial t^2}+\beta(\mathrm{t})\frac{\partial u}{\partial t}-\frac{\partial^2 u}{\partial x^2}=0, \qquad (1)$$

where $\beta(\mathrm{t})$ will be defined as

$$\beta(\mathrm{t})=\begin{cases} A, & 0<t<T_0 \\ 0, & T_0<t<T, \end{cases}$$

A is a real constant. Boundary conditions will have a form of

$$u(t,0)=0,\ \ u(t,\pi)=0,\ \ 0\le t\le T. \qquad (2)$$

Here we will impose initial conditions

$$u(0,x)=\varphi(x),\ 0\le x\le\pi,\ u_t(0,x)=0, 0<x<\pi. \qquad (3)$$

The domain of consideration is a rectangle $\Omega=\{(t,x):0<x<\pi, 0<t<T\}$. Since $\beta(t)$ is changing its value depending on *t*, we need to divide Ω into two parts, namely,

$\Omega_1 = \{(t,x): 0 < x < \pi, 0 < t < T_0\}$ and $\Omega_2 = \{(t,x): 0 < x < \pi, \mathrm{T}_0 < t < T\}$. Therefore, equation (1) reads as

$$\frac{\partial^2 u}{\partial t^2} + A\frac{\partial u}{\partial t} - \frac{\partial^2 u}{\partial x^2} = 0, \quad 0 < t < T_0,$$

$$\frac{\partial^2 u}{\partial t^2} - \frac{\partial^2 u}{\partial x^2} = 0, \quad T_0 < t < T \ . \tag{4}$$

On the line t=T_0 one needs to consider conjugating conditions

$$u(T_0+0,x) = u(T_0+0,x), 0 \le x \le \pi, \tag{5}$$

$$\left.\frac{\partial u}{\partial t}\right|_{t=T_0-0} = \left.\frac{\partial u}{\partial t}\right|_{t=T_0+0}, 0 < x < \pi. \tag{6}$$

Our aim is to find a function $u(t,x)$, which satisfies equation (4) together with boundary conditions (2), initial conditions (3) and conjugation conditions (5)-(6). This function should also satisfy the following regularity conditions.

$$u(t,x) \in C(\bar{\Omega}) \cap C^2(\Omega_1 \cup \Omega_2) \cap C^1(\Omega). \tag{7}$$

We will search for nontrivial solution $u(t,x)$ as follows:

$$u(t,x) = u(t)\cdot X(x). \tag{8}$$

Substituting (8) into (4) and (2) we will get the following eigenvalue problem in x-variable:

$$\begin{cases} X''(x) + \lambda X(x) = 0 \\ X(0) = X(\mathrm{T}) = 0, \end{cases} \tag{9}$$

where λ is a real number.

It is well-known that the problem (9) has nontrivial solutions at $\lambda = n^2$, n=1,2,... [1]

$$X_n(x) = \sin nx \ . \tag{10}$$

In time-variable we will get the following equations:

$$u'' + Au' + \lambda u = 0, 0 < t < T_0, \tag{11}$$

$$u'' + \lambda u = 0, T_0 < t < T. \tag{12}$$

General solution of (11) and (12) have a form

$$u(t) = e^{-\frac{A}{2}t}\left(C_1 \cos\frac{\sqrt{4\lambda - A^2}}{2}t + C_2 \sin\frac{\sqrt{4\lambda - A^2}}{2}t\right), \qquad -2\sqrt{\lambda} < A < 2\sqrt{\lambda},$$

$$u(t) = C_3 \cos\sqrt{\lambda}t + C_4 \sin\sqrt{\lambda}t .$$

Considering that $\lambda = n^2$, we have

$$u_n(t) = e^{-\frac{A}{2}t}\left(\mathrm{C}_{1n} \cos\frac{\sqrt{4\lambda - A^2}}{2}t + C_{2n} \sin\frac{\sqrt{4\lambda - A^2}}{2}t\right), \quad -2 < A < 2, \tag{13}$$

$$u_n(t) = C_{3n}\cos nt + C_{4n}\sin nt \ . \tag{14}$$

According to the superposition principle, we have

$$u(t,x)=\sum_{n=1}^{\infty}u_n(t)\sin nx, \qquad (15)$$

where $u_n(t)$ is defined by (13) for $0<t<T_0$ and by (14) for $\mathrm{T}_0<t<T$.

In order to find unknown constants C_{in} $\left(i=\overline{1,4}\right)$, we will use initial conditions (3) and conjugating conditions (5)-(6).

After the finding coefficients C_{in}, we need to prove the uniform convergence of series (15) and other infinite series corresponding to functions $u_{tt}(t,x), u_{xx}(t,x)$. During this process we have to impose certain conditions to given data in order to guarantee that found function $u(t,x)$ will satisfy regularity conditions (7).

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