CONSTRUCTION OF OPTIMAL QUADRATURE FORMULAS IN SOBOLEV SPACE.

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Abstract: *In this article, a series of problems related to the creation and application of quadrature and cubature formulas, including: finding errors of quadrature and cubature formulas in the Gilbert phases of differential functions; calculating error functionals of found extremal functions using extremal functions; finding the conditions for the existence and uniqueness of optimal quadrature and cubature formulas; developing new algorithms for constructing discrete operators based on optimal quadrature formulas, as well as determining the optimal coefficient adjustments, are discussed.*

Keywords: *Integral , differential , equations , algebraic , variational approaches , interpolation formulas , polynomials , mathematical analysis , asymptotic , optimal, cubature , assumption ,*

INTRODUCTION

In the context of numerous scientific and practical studies conducted globally, solutions to problems arising are presented in terms of integral and differential equations. They are primarily solved using cubature and interpolation formulas. The algebraic and variational approaches to constructing such formulas exist, with initial algebraic formulas including Newton-Cotes, Gauss-type quadrature formulas, as well as Lagrange and Newton interpolation polynomials. The theory of variational approaches to creating such formulas is based on the work of American and Russian scientists. Developing new algorithms for constructing optimal formulas and interpolation splines based on algebraic and variational approaches, as well as evaluating their errors, is an essential task in mathematical analysis. In the years of independence in our country, attention has been paid to practical applications, especially emphasizing the theory of cubature formulas in mathematical analysis with a high degree of algebraic precision. Special attention has been given to creating Gauss-type cubature formulas based on the theory of invariants and orthogonal polynomials relative to the group of reflections of a regular simplex, which has led to significant results. Optimal cubature formulas for continuous and non-continuous functions with one or more variables in Sobolev spaces have been developed, achieving remarkable results.

THE MAIN PART

At present, mathematical models with high precision for natural processes, formulated as differential and integral equations, as well as their systems, are gaining significant importance. The development of optimal quadrature and cubature formulas, as well as interpolation splines, in the Gilbert phases of differential functions for approximate solutions of these models is crucial. Purposeful scientific research, particularly in the following directions, is considered one of the important tasks: creating asymptotic optimal cubature formulas for various Gilbert and Banach phases of continuous and non-continuous functions; developing cubature formulas based on Monte Carlo methods; constructing optimal quadrature and cubature formulas and evaluating their errors; and creating splines that minimize specific functionals. The scientific research conducted in the aforementioned directions within the scope of the stated scientific investigations elucidates the relevance of the topic of this dissertation.

OPTIMUM QUADRATURE FOR APPROXIMATE CALCULATION OF FOURIER INTEGRALS FORMULAS

$\frac{d^4}{dx^4}-2\frac{d^2}{dx^2}+1$ **The discrete analogue of the operator.**

The structure of differential operators discretized with discrete analogs of optimal quadrature and interpolation formulas depends on various Gilbert phases,the analytical algorithm for finding coefficients in the $L_2^{(m)}$ phases of optimal cubature formulas is presented in the work [9,10]. S.L. Sobolev determined the discrete analogue of the Δ^m polyharmonic operator $D_{hH}^{(m)}[\beta]$ and studied the constructed discrete analog parameters.The discrete operator $D_{hH}^{(m)}[\beta]$ is quite complex to construct for n-dimensional cases, and this problem is still open. For a one-dimensional case $\frac{d^{2m}}{dx^{2m}}$, the discrete analogue of the differential operator $D_h^{(m)}[h\beta]$ was constructed by Z. J. Jamolov and X. M. Shadimetov.

$\frac{d^4}{dx^4}-2\frac{d^2}{dx^2}+1$ In this place, we engage in constructing the discrete analogue of the operator. Let's examine the following equation.

$$D(h\beta)*G(h\beta)=\delta(h\beta), \qquad (1.1)$$

In this place, $G(h\beta)$- corresponds to the discrete argument function that is suitable for the following function

$$G(x)=\frac{\operatorname{sgn}x}{4}\left(x\frac{e^x+e^{-x}}{2}-\frac{e^x-e^{-x}}{2}\right),$$

(1.2)

In this place, $\delta(h\beta)$ is the discrete delta function.

The search for a function *D(hβ)* satisfying equation (1.1) with a discrete argument is necessary.

CERTAIN FORMULAS

Here, we mainly rely on discrete argument functions and their operations [1,2].

Now, let's illustrate several known formulas used in constructing the discrete analogue of the differential operator (for example, see [1]).

For continuous functions, direct and inverse Fourier transformations are correct.

$$F[\varphi] = \int_{-\infty}^{\infty} \varphi(x) e^{2\pi ipx} dx, F^{-1}[\varphi] = \int_{-\infty}^{\infty} \varphi(p) e^{-2\pi ipx} dp,$$

(1.3)

Distinctive properties of Fourier transformations.

$$F[\varphi * \psi] = F[\varphi] \cdot F[\psi],$$

(1.4)

$$F[\varphi \cdot \psi] = F[\varphi] * F[\psi],$$

(1.5)

$$F[\delta^{(a)}(x)] = (-2\pi ip)^a, F[\delta(x)] = 1.$$

(1.6)

δ function properties,

$$\delta(hx) = h^{-1}\delta(x)$$

(1.7)

$$\delta(x-a) \cdot f(x) = \delta(x-a) \cdot f(a),$$

(1.8)

$$\delta^{(a)}(x) * f(x) = f^{(a)}(x),$$

(1.9)

$$\phi_0(x) = \sum_{\beta=-\infty}^{\infty} \delta(x-\beta) = \sum_{\beta=-\infty}^{\infty} e^{2\pi ix\beta}$$

(1.10)

We now use the solution of the system (1.22)-(1.24) with the assumption that *β<0* and *β>N*. Let $C_\beta = 0$ be defined as . Then, utilizing the definitions, we can express the system (1.22)-(1.24) in the following form in a compact manner.

$$C_\beta * G(h\beta) + d_1 e^{h\beta} + d_2 e^{-h\beta} = f(h\beta), \beta = 0,1,...N,$$

(1.11)

$$\sum_{\beta=0}^{N} C_\beta e^{h\beta} = -1,$$

(1.12)

$$\sum_{\beta=0}^{N} C_\beta e^{-h\beta} = (1-e^{-1}),$$

(1.13)

In this place , $f(h\beta)\int_0^1 G(x-h\beta)dx =$

$$= \frac{1}{8}[8 + (1+e^{-1})(h\beta)e^{h\beta} - (1+e)(h\beta)e^{-h\beta} - (2+3e^{-1})e^{h\beta} - (2+e)e^{-h\beta}].$$

(1.14)

Now we solve the problem equivalent to the following problem 2.

Problem 1. Given $f(h\beta)$, find constants for $C_\beta, \beta = 0,1,...,N$ and d_1, d_2 to satisfy the system (1.11)-(1.13) .Afterwards, we express C_β in terms of the following functions $v(h\beta)$ and $u(h\beta)$.

$$v(h\beta) = C_\beta * G(h\beta),$$

(1.15)

$$u(h\beta) = v(h\beta) + d_1 e^{h\beta} + d_2 e^{-h\beta}.$$

(1.16)

In this case, we express the coefficients C_β through the function $u(h\beta)$. For this purpose, we utilize the discrete argument function *D(hβ)* satisfying the (1.1) equation, namely, using Theorems 4 and 5. Consequently, by considering equation (1.16) and taking into account Theorems 4 and 5, we obtain the following expression:

$$C_\beta = D(h\beta) * u(h\beta)$$

(1.17)

In this way, finding the function $u(h\beta)$, the optimal coefficients for C_β can be obtained from equation (1.17). To compute the derivative in equation (1.17), it is required to have all values of the function $u(h\beta)$ for every integer β Based on equation (1.11), where $\beta = 0,1,...,N,$ we need the values of $u(h\beta) = f(h\beta)$ for all integers in this range. Now, for $\beta < 0$ and $\beta > N$, the form of the function $u(h\beta)$ is needed. Given that $C_\beta = 0$ for $\beta < 0$ and $\beta > N$, the expression of $u(h\beta)$ should accommodate these conditions.

$$C_\beta = D(h\beta) * u(h\beta) = 0, h\beta \notin [0,1]$$

Let's assume, for instance, that $\beta < 0$ In this case, we can use equations (1.11) and the form of the function $G(x)$ as follows.

$$v(h\beta) = G(h\beta) * C_\beta = \sum_{\gamma=-\infty}^{\infty} C_\gamma G(h\beta - h\gamma) = \sum_{\gamma=0}^{N} C_\gamma G(h\beta - h\gamma) =$$

$$= \sum_{\gamma=0}^{N} C_\gamma \frac{\operatorname{sgn}(h\beta - h\gamma)}{4}\left((h\beta - h\gamma)\frac{e^{h\beta - h\gamma} + e^{h\gamma - h\beta}}{2} - \frac{e^{h\beta - h\gamma} - e^{h\gamma - h\beta}}{2}\right) =$$

$$=-\frac{1}{8}\sum_{\gamma=0}^{N}C_{\gamma}\left[(h\beta-h\gamma)e^{h\beta-h\gamma}+(h\beta-h\gamma)e^{h\gamma-h\beta}-e^{h\beta-h\gamma}+e^{h\gamma-h\beta}\right]=$$

$$=-\frac{1}{8}\sum_{\gamma=0}^{N}C_{\gamma}\left[h\beta\cdot e^{h\beta-h\gamma}-h\gamma\cdot e^{h\beta-h\gamma}+h\beta\cdot e^{h\gamma-h\beta}-h\gamma\cdot e^{h\gamma-h\beta}-e^{h\beta-h\gamma}+e^{h\gamma-h\beta}\right]=$$

$$=-\frac{1}{8}\sum_{\gamma=0}^{N}C_{\gamma}\left[h\beta\cdot e^{h\beta}\cdot e^{-h\gamma}+h\beta\cdot e^{h\gamma}\cdot e^{-h\beta}-e^{h\beta}\cdot e^{h\gamma}+e^{h\gamma}\cdot e^{-h\beta}-h\gamma\cdot e^{h\beta}\cdot e^{-h\gamma}+h\gamma\cdot e^{-h\beta}\cdot e^{h\gamma}\right]=$$

$$=-\frac{1}{8}\sum_{\gamma=0}^{N}C_{\gamma}\left[(h\beta\cdot e^{h\beta}-e^{h\beta})\cdot e^{-h\gamma}+(h\beta\cdot e^{-h\beta}+e^{-h\beta})\cdot e^{h\gamma}-h\gamma\cdot e^{h\beta}\cdot e^{-h\gamma}+h\gamma\cdot e^{-h\beta}\cdot e^{h\gamma}\right]=$$

$$=-\frac{1}{8}\sum_{\gamma=0}^{N}C_{\gamma}\left[(h\beta\cdot e^{h\beta}-e^{h\beta})\cdot\sum_{\gamma=0}^{N}C_{\gamma}e^{-h\gamma}+(h\beta\cdot e^{-h\beta}+e^{-h\beta})\cdot\sum_{\gamma=0}^{N}C_{\gamma}e^{h\gamma}-e^{h\beta}\cdot\sum_{\gamma=0}^{N}C_{\gamma}\cdot h\gamma\cdot e^{-h\gamma}+e^{-h\beta}\sum_{\gamma=0}^{N}C_{\gamma}\cdot h\gamma\cdot e^{h\gamma}\right]$$

taking equations (1.11) and (1.12) into account,

$$v(h\beta)=-\frac{1}{8}\left[(h\beta-1)e^{h\beta}\cdot(1-e^{-1})+(h\beta+1)e^{-h\beta}\cdot(e-1)-e^{h\beta}\cdot\sum_{\gamma=0}^{N}C_{\gamma}\cdot h\gamma\cdot e^{-h\gamma}+e^{-h\beta}\sum_{\gamma=0}^{N}C_{\gamma}\cdot h\gamma\cdot e^{h\gamma}\right]=$$

$$=-\frac{1}{8}\left[(h\beta-1)e^{h\beta}\cdot(1-e^{-1})+(h\beta+1)e^{-h\beta}\cdot(e-1)\right]+$$

$$+\frac{1}{8}e^{h\beta}\cdot\sum_{\gamma=0}^{N}C_{\gamma}\cdot h\gamma\cdot e^{-h\gamma}-\frac{1}{8}e^{-h\beta}\sum_{\gamma=0}^{N}C_{\gamma}\cdot h\gamma\cdot e^{h\gamma}.$$

$b_1=\frac{1}{8}\sum_{\gamma=0}^{N}C_{\gamma}\cdot h\gamma\cdot e^{-h\gamma}, b_2=\frac{1}{8}\sum_{\gamma=0}^{N}C_{\gamma}\cdot h\gamma\cdot e^{h\gamma}$ specifying that, $\beta<0$ at we get the following

$$v(h\beta)=-\frac{1}{8}\left[(h\beta-1)e^{h\beta}\cdot(1-e^{-1})+(h\beta+1)e^{-h\beta}\cdot(e-1)\right]-b_1e^{h\beta}-b_2e^{-h\beta}$$

(1.18)

And now $\beta>N$

$$v(h\beta)=\frac{1}{8}\left[(h\beta-1)e^{h\beta}\cdot(1-e^{-1})+(h\beta+1)e^{-h\beta}\cdot(e-1)\right]-b_1e^{h\beta}-b_2e^{-h\beta}$$

(1.19)

$$u(h\beta)=\begin{cases}-\frac{1}{8}\left[(h\beta-1)e^{h\beta}\cdot(1-e^{-1})+(h\beta+1)e^{-h\beta}\cdot(e-1)\right]+ \\ \qquad +b_1e^{h\beta}-b_2e^{-h\beta}+d_1e^{h\beta}+d_2e^{-h\beta}, \beta\le 0 \\ f(h\beta), \qquad 0\le\beta\le N \\ \frac{1}{8}\left[(h\beta-1)e^{h\beta}\cdot(1-e^{-1})+(h\beta+1)e^{-h\beta}\cdot(e-1)\right]- \\ \qquad -b_1e^{h\beta}+b_2e^{-h\beta}+d_1e^{h\beta}+d_2e^{-h\beta}, \beta\ge N\end{cases}$$

$$d_1^- = b_1 + d_1, \qquad d_2^- = -b_2 + d_2,$$

$$d_1^+ = b_1 + d_1, \qquad d_2^+ = -b_2 + d_2,$$

we come to the following issue by entering the designation.

Conclusion

In this work, an exponential optimal quadrature formula has been constructed in the Gilbert space with a dimension of $K_2(P_2)$. Initially, an extremal function was found to calculate the norm of the error functional for this quadrature formula. Then, using this extremal function, the norm of the error functional was determined. The existence and convergence of the exponential optimal quadrature formula were demonstrated. A system of linear equations was obtained for the coefficients of the optimal quadrature formula. To find an analytical representation of the optimal coefficients that will be the solution of this system $\frac{d^4}{dx^4}-2\frac{d^2}{dx^2}+1$ differential operator a discrete analog was built and its properties were studied. Finally, optimal odds found.

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